University of Colorado
Department of Mathematics

2008/2009 Semester 2 | Math 6360 Complex Variables 2 | Assignment 4

Due Monday April 13, 2009

1. Read Ahlfors Chapter 5, Sections 5.5.1 – 5.5.5, skimming proofs. Do Problem 2 on p. 227.
2. Read Ahlfors Chapter 6, Sections 1, 2 and 3. Do problem 1, p. 232, and problems 3 and 4, p. 238.
3. Fix $a, b, c, d \in \mathbb{R}$ with $0 < a < b$ and $0 < c < d < 2\pi$. Show that the map $f(z) = e^z$ maps the open rectangle $\{z = x + iy \in \mathbb{C} : \ a < x < b,\ c < y < d\}$ onto the region $\{w = re^{i\phi} : \ e^a < r < e^b,\ c < \phi < d\}$. Is this map conformal? Is it one-to-one?
4. Consider the function $g(z) = \sin z$ applied to the semi-infinite strip $\{z = x + iy : -\frac{\pi}{2} \leq \ x \ \leq \frac{\pi}{2},\ y \geq 0\}$. Prove that the image of this infinite strip is the closed upper half plane $\{w = x + iy : \ y \geq 0\}$.
5. (a) Let $x_1,\ x_2,\ x_3 \in \mathbb{R}$, with $x_1 < x_2 < x_3$, and suppose that we consider the triangle in $\mathbb{C}$ with distinct vertices $w_1,\ w_2,\ w_3 \in \mathbb{C}$ having respective exterior angles $\beta_1\pi,\ \beta_2\pi$, and $\beta_3\pi$, where $-1 < \beta_k < 1,\ 1 \leq k \leq 3$. Show that in this case a variation of the Schwarz-Christoffel transformation from the upper half plane onto the triangle that carries x_k to $w_k,\ 1 \leq k \leq 3$, is given by

   $$F(z) = C \int_{z_0}^{z} (s - x_1)^{-\beta_1}(s - x_2)^{-\beta_2}(s - x_3)^{-\beta_3} ds \ + C',$$

   where $z_0,\ C$ and C' are chosen appropriately in $\mathbb{C}$.

   (b) In part (a) above, let $C = e^{3\pi i/4},\ z_0 = C' = 0,\ x_1 = -1,\ x_2 = 0,\ x_3 = 1,\ \beta_1 = \frac{3}{4},\ \beta_2 = \frac{1}{2},\ \beta_3 = \frac{3}{4}$. Explicitly describe the image of the upper half plane $\overline{H} = \{z : \ \mathrm{Im}(z) \geq 0\}$ under this transformation.