University of Colorado
Department of Mathematics

2016/2017 Semester 2 Math 8340 Functional Analysis 2 Assignment 3

Some solutions

Q 2 Here we consider $\ell^1(\mathbb{Z})$. For $j \in \mathbb{Z}$, let δ_j be the function on $\mathbb{Z}$ that is 1 if $n = j$, and is 0 for $n \neq j$. Note $\delta_j \in \ell^1(\mathbb{Z})$, $\forall j \in \mathbb{Z}$, and if $\lambda \in \mathbb{C}$,

$$[\lambda\delta_j]^* = \overline{\lambda}\delta_{-j}.$$

Now consider $f \in \ell^1(\mathbb{Z})$ defined by

$$f(n) = \delta_{-1}(n) + \delta_0(n) + i\delta_1(n).$$

Note

$$\|f\| = 1 + 1 + 1 = 3, \text{ and } \|f\|^2 = 9.$$

An easy calculation shows

$$f^*(n) = -i\delta_{-1}(n) + \delta_0(n) + \delta_1(n),$$

and

$$(f^*) * f(n) = -i\delta_{-2}(n) + (1-i)\delta_{-1}(n) + 3\delta_0(n) + (1+i)\delta_1(n) + i\delta_2(n).$$

Therefore

$$\|(f^*) * f\| = 5 + 2\sqrt{2} < 5 + 2 \cdot 2 = 9 = \|f\|^2,$$

so that $\ell^1(\mathbb{Z})$ is not a C^*-algebra.

Q 5 (a) Let $\mathcal{C}$ be the C^*-subalgebra of $\mathcal{A}$ generated by $\{1, b, b^{-1}\}$. Note that $\mathcal{C}$ is commutative. Consider $G : \mathcal{C} \to C(\Sigma(\mathcal{C}))$, where $\Sigma := \Sigma(\mathcal{C})$ is the maximal ideal space of $\mathcal{C}$ and G is the Gelfand transform. By the Gelfand-Naimark Theorem, G is an isometric $*$-isomorphism. Hence for all $a \in \mathcal{C}$, $\|a\|_{\mathcal{C}} = \sup_{\phi\in\Sigma}\{|\hat{a}(\phi)|\}$. Since b is self adjoint, $\sigma_{\mathcal{C}}(b)$ is contained in $\mathbb{R}$, and since b is invertible, $0 \notin \sigma_{\mathcal{C}}(b)$ so that there exist $\delta, M > 0$ with $\sigma_{\mathcal{C}}(b) \subseteq [-M, -\delta] \cup [\delta, M]$. The function $g(t) = \frac{1}{t}$ is continuous on $[-M, -\delta] \cup [\delta, M]$, so by a variant of Weierstrass' polynomial approximation theorem, there exists a sequence of polynomials in t, $\{p_n(t)\}$, such that $\{p_n(t)\}$ converges to $\frac{1}{t}$ uniformly on $[-M, -\delta] \cup [\delta, M]$. Thus given $\epsilon > 0$, there exists $N > 0$ and $\epsilon' > 0$ such that whenever $n \geq N$,

$$|p_n(t) - \frac{1}{t}| < \epsilon' < \epsilon, \ \forall t \in [-M, -\delta] \cup [\delta, M].$$

But for every $\phi \in \Delta$, $\phi(b) = \hat{b}(\phi) \in \sigma_{\mathcal{C}}(b) \subseteq [-M, -\delta] \cup [\delta, M]$. Thus whenever $n \geq N$,

$$|p_n(\hat{b}(\phi)) - \frac{1}{\hat{b}(\phi)}| < \epsilon' < \epsilon, \ \forall \phi \in \Sigma.$$

Now by properties of the Gelfand transform, $p_n(\hat{b}(\phi) = \widehat{p_n(b)}(\phi)$ and $\frac{1}{\hat{b}(\phi)} = \widehat{b^{-1}}(\phi)$, so that we have whenever $n \geq N$,

$$|\widehat{p_n(b)}(\phi) - \widehat{b^{-1}}(\phi)| = |[p_n\widehat{(b) - b}^{-1}](\phi)| < \epsilon' < \epsilon, \ \forall \phi \in \Sigma.$$

Thus $\sup_{\phi \in \Sigma} |[p_n\widehat{(b) - b}^{-1}](\phi)| \leq \epsilon' < \epsilon$, for all $n \geq N$. It follows by our earlier remarks that $\|p_n(b) - b^{-1}\|_{\mathcal{C}} = \|p_n(b) - b^{-1}\|_{\mathcal{A}} < \epsilon$ whenever $n \geq N$. Since ϵ was arbitrary, this implies that b^{-1} is in the C^*-algebra generated by 1 and b, hence $b^{-1} \in \mathcal{B}$, as we desired to show.