University of Colorado
Department of Mathematics

2008/2009 Semester 2 — Math 4330/5330 Fourier Analysis — Assignment 10

Selected Solutions

6.2.2 As the hint suggests, we use Proposition 6.2.1(c) and Example 6.1.1.

$$\mathcal{F}(x\chi_{[-1,1]}(x))(s) \;=\; (-2\pi i)^{-1}[\mathcal{F}(\chi_{[-1,1]}(x))]'(s)$$

$$= \;(-2\pi i)^{-1}[\frac{\sin 2\pi s}{\pi s}]' \;=\; (-2\pi i)^{-1}\frac{2\pi\cos 2\pi s(\pi s) - \pi\sin 2\pi s}{\pi^2 s^2}$$

$$= \;\frac{i}{2}\frac{(2\pi s\cos 2\pi s - \sin 2\pi s)}{\pi^2 s^2}.$$

6.2.3 By Exercise 5.2.2,

$$f(x) \;=\; (2b - |x|)\chi_{[-2b,2b]}(x) \;=\; \chi_{[-b,b]} * \chi_{[-b,b]}(s).$$

So

$$\mathcal{F}(f)(s) \;=\; \mathcal{F}(\chi_{[-b,b]} * \chi_{[-b,b]})(s)$$

and by Proposition 6.2.1 (e) this is equal to:

$$\mathcal{F}(\chi_{[-b,b]})(s)\cdot\mathcal{F}(\chi_{[-b,b]})(s) \;=\; (2b\mathrm{sinc}(2\pi bs))\cdot(2b\mathrm{sinc}(2\pi bs))$$

$$= \;4b^2(\mathrm{sinc}(2\pi bs))^2.$$

6.2.6 (For Math 5330 students only.) Recall $\Gamma(p) \;=\; \int_0^\infty x^{p-1}e^{-x}dx$, an integral which converges for all $p > 0$. We are given that $\Gamma(1/2) = \sqrt{\pi}$, and we want to show that

$$\int_{-\infty}^{\infty} e^{-\pi x^2}dx \;=\; 1.$$

We start with the given information and use the hint:

$$\sqrt{\pi} \;=\; \Gamma(1/2) \;=\; \int_0^\infty x^{-1/2}e^{-x}dx \;=\; \int_0^\infty \frac{1}{\sqrt{x}}e^{-x}dx.$$

We now substitute $u = \sqrt{x/pi}$. Then $\pi u^2 = x$, and $du = \frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{x}}dx$, so that

$$\frac{1}{\sqrt{x}}dx \;=\; 2\sqrt{\pi}du.$$

It follows that

$$\int_0^\infty \frac{1}{\sqrt{x}}e^{-x}dx \;=\; \int_0^\infty 2\sqrt{\pi}e^{-\pi u^2}du.$$

Thus

$$\sqrt{\pi} \;=\; \int_0^\infty \frac{1}{\sqrt{x}} e^{-x} dx \;=\; 2\sqrt{\pi} \int_0^\infty e^{-\pi u^2} du.$$

This gives

$$\frac{1}{2} \;=\; \int_0^\infty e^{-\pi u^2} du,$$

and since $u \to e^{-\pi u^2}$ is even as a function of u, we get

$$1 \;=\; \int_{-\infty}^\infty e^{-\pi u^2} du.$$

But changing back to the variable x gives

$$1 \;=\; \int_{-\infty}^\infty e^{-\pi x^2} dx,$$

as we desired to show.

6.4.1 Here $f(x) = \frac{1}{1+(ax+b)^2}$. Let $D_a(g)(x) = g(ax)$, and let $T_b(h)(x) = h(x-b)$. By Prop 6.2.1 (b)(i), we know that

$$\mathcal{F}(D_a(g)(x))(s) \;=\; \frac{1}{|a|} \mathcal{F}(g(x))(\frac{s}{a}) = \frac{1}{|a|} D_{1/a} \mathcal{F}(g)(s).$$

By Proposition 6.2.1 (a)(ii), we know that

$$\mathcal{F}(T_{-b}(h)(x))(s) \;=\; e^{2\pi i s b} \mathcal{F}(h)(s).$$

One easily calculates that setting $g(x) = \frac{1}{1+x^2}$,

$$f(x) = \frac{1}{1+(ax+b)^2} \;=\; (T_{-b}g)(ax) \;=\; D_a T_{-b} g(x).$$

It follows that

$$\mathcal{F}(f)(x) \;=\; \mathcal{F}(D_a T_{-b} g(x))(s) \;=\; \frac{1}{|a|} D_{1/a} \mathcal{F}(T_{-b} g(x))(s)$$

$$= \frac{1}{|a|} D_{1/a} [e^{2\pi i s b} \mathcal{F}(g)](s) \;=\; \frac{1}{|a|} D_{1/a} [e^{2\pi i s b} \pi e^{-2\pi |s|}]$$

$$= \frac{\pi}{|a|} e^{\frac{2\pi i s b}{a}} e^{-2\pi |\frac{s}{a}|}.$$

6.5.1 Let $h(x) \;=\; (\text{sinc } \pi x)^2$. Then$h(x) = 1$ for $x = 0$ and $h(x) = \frac{[\sin \pi x]^2}{\pi^2 x^2}$ for $x \neq 0$. Then

$$\hat{h}(s) \;= \mathcal{F}(h)(s) \;=\; \mathcal{F}((\text{sinc } \pi x) \cdot (\text{sinc } \pi x))(s)$$

$$= \mathcal{F}((\text{sinc } \pi x)) * \mathcal{F}((\text{sinc } \pi x))(-s)$$

(by Proposition 6.2.1(e))

$$\chi_{[-\frac{1}{2},-\frac{1}{2}]} * \chi_{[-\frac{1}{2},-\frac{1}{2}]}(-s) \;=\; (1-|(-s)|)\chi_{[-1,1]}(-s)$$

$$= \; (1-|s|)\chi_{[-1,1]}(s).$$

6.5.5 We let $f(x) = 1/(1+x^2)$ and $g(x) = e^{-4\pi i x}\text{sinc}(6\pi x)) = e^{-2(2\pi i x)}\text{sinc}(2\pi 3x))$. Recall that $\hat{f}(s) = \pi e^{-2\pi|s|}$ and

$$\mathcal{F}(e^{-2(2\pi i x)}\text{sinc}(2\pi 3x)))(s) \;=\; \mathcal{F}(\text{sinc}(2\pi 3x)))(s+2).$$

Now

$$\mathcal{F}(\text{sinc}(2\pi 3x)))(s) \;=\; \mathcal{F}\mathcal{F}(\frac{1}{6}\chi_{[-3,3]})(x))(s)$$

$$= \; \frac{1}{6}\chi_{[-3,3]}(-s) \;=\; \frac{1}{6}\chi_{[-3,3]}(s).$$

It follows that

$$\mathcal{F}(\text{sinc}(2\pi 3x)))(s+2) \;=\; \frac{1}{6}\chi_{[-3,3]}(s+2) \;=\; \frac{1}{6}\chi_{[-5,1]}(s).$$

Hence by the Plancherel Theorem,

$$\langle f, g\rangle \;=\; \langle \hat{f}, \hat{g}\rangle \;=\; \langle \pi e^{-2\pi|s|}, \frac{1}{6}\chi_{[-5,1]}(s)\rangle$$

$$= \; \int_{-\infty}^{\infty} \pi e^{-2\pi|s|}\overline{\frac{1}{6}\chi_{[-5,1]}(s)}ds$$

$$= \; \frac{\pi}{6}\int_{-\infty}^{\infty} e^{-2\pi|s|}\chi_{[-5,1]}(s)ds$$

$$= \; \frac{\pi}{6}\int_{-5}^{0} e^{2\pi s}ds + \frac{\pi}{6}\int_{0}^{1} e^{-2\pi s}ds \;=\; [\frac{\pi}{2\pi\cdot 6}e^{2\pi s}]_{s=-5}^{0} + [\frac{\pi}{-2\pi\cdot 6}e^{-2\pi s}]_{s=0}^{1}$$

$$= \; \frac{1}{12}[1-e^{-10\pi}] - \frac{1}{12}[e^{-2\pi}-1] \;=\; \frac{1}{12}[2-e^{-10\pi}-e^{-2\pi}].$$