University of Colorado
Department of Mathematics

2008/2009 Semester 2 Math 4330/5330 Fourier Analysis Assignment 11

Selected Solutions

6.5.2 If $h(x) = (\text{sinc } \pi x)^2$, then since $h(x) = 1$ for $x = 0$ and $h(x) = \frac{[\sin \pi x]^2}{\pi^2 x^2}$ for $x \neq 0$, the quotient rule tells us that

$$h'(x) = \frac{[2\pi \sin \pi x \cos \pi x] \cdot [\pi^2 x^2] - [2\pi^2 x] \cdot [\sin \pi x]^2}{\pi^4 x^4}$$

$$= \frac{2\pi x \sin \pi x \cos \pi x - 2(\sin \pi x)^2}{\pi^2 x^3} = 2\frac{\pi x \sin \pi x \cos \pi x - (\sin \pi x)^2}{\pi^2 x^3}$$

$$= \frac{2}{\pi^2} \frac{\pi x \sin \pi x \cos \pi x - (\sin \pi x)^2}{x^3} = \frac{2}{\pi^2} k(x), \ x \neq 0,$$

where $k(x)$ is as in the statement of the problem, and $h'(0)$ is some non-zero constant.

It follows that

$$k(x) = \frac{\pi^2}{2} h'(x),$$

so that

$$\hat{k}(s) = \frac{\pi^2}{2} \widehat{h'}(s).$$

We have seen in Problem 6.5.1 assigned in HW 10 that

$$\hat{h}(s) = (1 - |s|)\chi_{[-1,1]}(s).$$

Since h is piecewise smooth, continuous, and in $L^1(\mathbb{R})$, Propositions 6.2.1 c(i) tells us that

$$\mathcal{F}(h'(x))(s) = 2\pi i s \hat{h}(s),$$

as long as h' is in $L^1(\mathbb{R})$. But simple estimates show that for large x, the function $h'(x)$ grows at a rate proportional to $\frac{1}{x^2}$, so that $h' \in L^1(\mathbb{R})$.

Thus

$$\mathcal{F}(h'(x))(s) = 2\pi i s \hat{h}(s) = 2\pi i s (1 - |s|)\chi_{[-1,1]}(s).$$

It follows that

$$\hat{k}(s) = \frac{\pi^2}{2} \widehat{h'}(s) = \pi^3 i s (1 - |s|)\chi_{[-1,1]}(s).$$

6.5.3 (For Math 5330 students only.) We assume that f and g are in $C(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\hat{f}(s) = \hat{g}(s)$ a.e. Then $\hat{f}$ and $\hat{g}$ can be identified as elements in $L^2(\mathbb{R})$. It follows from Theorem 6.5.2, on p. 326, that

$$f(x) = \int_{\infty}^{\infty} \hat{f}(s) e^{2\pi i s x} dx = \int_{\infty}^{\infty} \hat{f}(s) e^{2\pi i s x} dx = g(x),$$

where, to quote this theorem, the integrals in the center are " understood to denote the mean square norm limit of the corresponding integrals over $[-N, N]$" and the equalities of the integrals with functions " mean that this limit is the element in $L^2(\mathbb{R})$" represented by f and g, respectively. Thus, what the equality above implies is exactly that $f = g$ as elements of $L^2(\mathbb{R})$. But this means that $f(x) = g(x)$ for almost all $x \in \mathbb{R}$. Let $E = \{x \in \mathbb{R} : f(x) = g(x)\}$. We claim that E contains a dense subset of $\mathbb{R}$. If not, then there exists a non-empty interval $I \subset E$ with $I \cap E = \emptyset$. But then for all $x \in I$, we have $f(x) \neq g(x)$, and this is a contradiction to the fact that $f(x) = g(x)$ a.e. in $\mathbb{R}$, because a non-empty interval I is certainly not a non-empty set. It follows that we can find a set $D \subset E$ with D dense in $\mathbb{R}$. Now let x_0 be an arbitrary element of $\mathbb{R}$. Since D is dense in $\mathbb{R}$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ with $\lim_{n\to\infty} x_n = x_0$. Since $D \subset E$, we have $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$. Now we use the continuity of f and g to deduce that:

$$f(x_0) \;=\; \lim_{n\to\infty} f(x_n) \;=\; \lim_{n\to\infty} g(x_n) \;=\; g(x_0).$$

We thus obtain $f(x_0) = g(x_0)$, and since x_0 was arbitrarily chosen, we get $f(x) = g(x)$ for all $x \in \mathbb{R}$.

7.4.1 Let $f(t) = \operatorname{sinc}(2\pi\beta\Omega t)$, $\beta \in (0, 1)$. By Example 6.5.1,

$$\hat{f}(s) \;= \mathcal{F}(f(t))(s) \;=\; (2\beta\Omega)^{-1}\chi_{[-\beta\Omega,\beta\Omega]}(s).$$

Since $0 < \beta\Omega < \Omega$, it follows that $\hat{f}$ is supported on $[-\Omega, \Omega]$, so that f is continuous and Ω-bandlimited. Thus, the Shannon sampling Theorem 7.4.1 applies and we can write

$$f(t) \;=\; \sum_{m=-\infty}^{\infty} f(\frac{m}{2\Omega})\operatorname{sinc}\pi(m - 2\Omega t),$$

that is,

$$\operatorname{sinc}(2\pi\beta\Omega t) \;=\; \sum_{m=-\infty}^{\infty} \operatorname{sinc}(\pi\beta m)\operatorname{sinc}\pi(m - 2\Omega t).$$

7.4.2 Please note that the problem as written does not quite make sense because of the $m = 0$ term in the summation. So we change the problem slightly. Let's try to calculate

$$\sum_{m\in\mathbb{Z}, m\neq 0} \frac{(-1)^m \sin \pi m\beta}{m(m-\beta)}$$

by using Exercise 7.4.1 above.
We choose $t = \frac{\beta}{2\Omega}$ in the last equality in Exercise 7.4.1. We then obtain:

$$\operatorname{sinc}(\pi\beta^2) \;=\; \sum_{m=-\infty}^{\infty} \operatorname{sinc}(\pi\beta m)\operatorname{sinc}\pi(m - \beta)$$

$$= \sum_{m\in\mathbb{Z},m\neq 0} \text{sinc}(\pi\beta m)\text{sinc}(\pi(m-\beta)) + \text{sinc}(-\pi\beta)$$

(by taking out the term $m = 0$ and recalling that $\text{sinc}(0) = 1$)

$$\sum_{m\in\mathbb{Z},m\neq 0} \frac{\sin(\pi\beta m)}{\pi\beta m} \cdot \frac{\sin(\pi(m-\beta))}{\pi(m-\beta)} + \text{sinc}(\pi\beta).$$

It follows that

$$\begin{aligned}
\text{sinc}(\pi\beta^2) - \text{sinc}(\pi\beta) &= \sum_{m\in\mathbb{Z},m\neq 0} \frac{\sin(\pi\beta m)}{\pi\beta m} \cdot \frac{\sin(\pi(m-\beta))}{\pi(m-\beta)} \\
&= \sum_{m\in\mathbb{Z},m\neq 0} \frac{\sin(\pi\beta m)\cdot(-1)^{m+1}\cdot\sin(\pi\beta)}{(\pi\beta m)\cdot(\pi(m-\beta))} \\
&= \frac{(-1)\sin(\pi\beta)}{\pi^2\beta} \sum_{m\in\mathbb{Z},m\neq 0} \frac{(-1)^m\sin(\pi\beta m)}{m(m-\beta)}.
\end{aligned}$$

We thus obtain

$$\text{sinc}(\pi\beta^2) - \text{sinc}(\pi\beta) = \frac{(-1)\sin(\pi\beta)}{\pi^2\beta} \sum_{m\in\mathbb{Z},m\neq 0} \frac{(-1)^m\sin(\pi\beta m)}{m(m-\beta)},$$

so that by using elementary algebra we get:

$$\sum_{m\in\mathbb{Z},m\neq 0} \frac{(-1)^m\sin(\pi\beta m)}{m(m-\beta)} = \frac{\pi^2\beta}{\sin(\pi\beta)}[\text{sinc}(\pi\beta) - \text{sinc}(\pi\beta^2)].$$

7.4.3 (For Math 5330 students only.)

(a) As on p. 376, we define

$$\sigma_m(t) = \text{sinc}\pi(m - 2\Omega t).$$

Since sinc is an even functions, we can write

$$\sigma_m(t) = \text{sinc}\pi(2\Omega t - m) = \text{sinc}2\pi\Omega(t - \frac{m}{2\Omega}).$$

Now let $\sigma(t) = \text{sinc}(2\pi\Omega t)$. By Example 6.5.1,

$$\hat{\sigma}(s) = \mathcal{F}(\sigma(t))(s) = (2\Omega)^{-1}\chi_{[-\Omega,\Omega]}(s).$$

Note that for each $m \in \mathbb{Z}$,

$$\sigma_m(t) = \sigma(t - \frac{m}{2\Omega}).$$

By Proposition 6.2.1 (a)(ii),

$$\mathcal{F}(\sigma_m(t))(s) \;=\; \mathcal{F}(\sigma(t-\frac{m}{2\Omega}))(s)$$

$$=\; e^{-\frac{2\pi ims}{2\Omega}}\mathcal{F}(\sigma(t))(s) \;=\; (2\Omega)^{-1}e^{-\frac{2\pi ims}{2\Omega}}\chi_{[-\Omega,\Omega]}(s).$$

By Parseval's Theorem, we see that

$$\|\sigma_m\|_{L^2(\mathbb{R})} \;=\; \|\widehat{\sigma_m}\|_{L^2(\mathbb{R})}$$

$$=\; \sqrt{\int_{-\infty}^{\infty}|(2\Omega)^{-1}e^{-\frac{2\pi ims}{2\Omega}}\chi_{[-\Omega,\Omega]}(s)|^2ds} \;=\; (2\Omega)^{-1}\sqrt{\int_{-\Omega}^{\Omega}1ds}$$

$$=\; \frac{1}{\sqrt{2\Omega}}.$$

We now show that the $\{\sigma_m\}_{m\in\mathbb{Z}}$ are orthogonal. By the Plancherel Theorem,

$$\langle\sigma_m,\sigma_n\rangle \;=\; \langle\widehat{\sigma_m},\widehat{\sigma_n}\rangle$$

$$=\; \int_{-\infty}^{\infty}(2\Omega)^{-1}e^{-\frac{2\pi ims}{2\Omega}}\chi_{[-\Omega,\Omega]}(s)\overline{(2\Omega)^{-1}e^{-\frac{2\pi ins}{2\Omega}}\chi_{[-\Omega,\Omega]}(s)}ds$$

$$=\; \int_{-\infty}^{\infty}(2\Omega)^{-2}e^{\frac{2\pi i(n-m)s}{2\Omega}}\chi_{[-\Omega,\Omega]}(s)ds = \;(2\Omega)^{-2}\int_{-\Omega}^{\Omega}e^{\frac{2\pi i(n-m)s}{2\Omega}}ds$$

which for $n\neq m$ is equal to

$$(2\Omega)^{-2}\frac{2\Omega}{2\pi i(n-m)}[e^{\frac{2\pi i(n-m)s}{2\Omega}}]_{s=-\Omega}^{s=\Omega} \;=\; \frac{1}{4\pi i\Omega(n-m)}[(-1)^{n-m}-(-1)^{m-n}] \;=\; 0.$$

It follows that the family of functions $\{\sigma_m\}_{m\in\mathbb{Z}}$ is an orthogonal family.

(b) We define the closed subspace $L^2(\Omega;\mathbb{R})$ by

$$L^2(\Omega;\mathbb{R}) \;=\; \{f\in L^2(\mathbb{R}):\ f \text{ is } \Omega-\text{bandlimited}\}.$$

We note that $\{\sigma_m\}_{m\in\mathbb{Z}}\subset\ L^2(\Omega;\mathbb{R})$ and we have shown that the $\{\sigma_m\}_{m\in\mathbb{Z}}$ are an orthogonal family. To show that the closed linear span of $\{\sigma_m\}_{m\in\mathbb{Z}}$ is equal to $L^2(\Omega;\mathbb{R})$ we consider the image of this family of functions and this subspace under the Fourier transform.

$$\{\mathcal{F}(\sigma_m)\}_{m\in\mathbb{Z}} \;=\; \{\widehat{\sigma_m}\}_{m\in\mathbb{Z}} \;=\; \{(2\Omega)^{-1}e^{-\frac{2\pi ims}{2\Omega}}\chi_{[-\Omega,\Omega]}(s)\}_{m\in\mathbb{Z}},$$

and

$$\mathcal{F}(L^2(\Omega;\mathbb{R})) \;=\; \{g\in L^2(\mathbb{R}):\ g \text{ is supported on } [-\Omega,\Omega]\}.$$

Since the Fourier transform gives an isomorphism of $L^2(\Omega;\mathbb{R})$ onto its range, if we can show that the linear span of $\{(2\Omega)^{-1}e^{-\frac{2\pi ims}{2\Omega}}\chi_{[-\Omega,\Omega]}(s)\}_{m\in\mathbb{Z}}$ is dense in $\{g \in L^2(\mathbb{R}) :\ g$ is supported on $[-\Omega, \Omega]\}$ in the L^2-norm, we will be done. But this is the same as showing that the linear span of $\{(2\Omega)^{-1}e^{-\frac{2\pi ims}{2\Omega}}\}_{m\in\mathbb{Z}}$ is dense in $L^2[-\Omega, \Omega]$ in the L^2 norm, and this follows from the ordinary theory of Fourier series for periodic functions defined on $\mathbb{R}$ with period $P = 2\Omega$; see for example, Section 1.9 and later, Theorem 3.6.2 (b) in the Stade textbook for the full statement of the needed result. Thus the closed linear span of $\{(2\Omega)^{-1}e^{-\frac{2\pi ims}{2\Omega}}\chi_{[-\Omega,\Omega]}(s)\}_{m\in\mathbb{Z}}$ in the L^2-norm is equal to $L^2[-\Omega, \Omega]$, so that the closed linear span of $\{\sigma_m\}_{m\in\mathbb{Z}}$ is equal to $L^2(\Omega; \mathbb{R})$.