University of Colorado
Department of Mathematics

2008/2009 Semester 2 | Math 5330 | Second Midterm Exam, Takehome

Due Wednesday April 8, 2009, 5 p.m. in Math 227.

This exam has two pages. No late papers will be accepted. You will not receive extra credit for doing the 4330 take-home exam.

INSTRUCTIONS: You are to work by yourself. You are allowed to use the textbook, class notes, previous homework assignments, the previous exam, and any other book you find helpful. If you need any clarification about a problem, you should consult me, and not other students.

1. (a) Prove that when $x \neq 0, \pm 2\pi, \pm 4\pi, \cdots$, the function

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos(nx)$$

can be expressed as

$$D_N(x) = \frac{\sin[(2N+1)x/2]}{2\sin(x/2)}.$$

(b) Prove that if $f : [-\pi, \pi] \to \mathbb{R}$ is piecewise continuous,

$$S_N^f(x) = \frac{a_0(f)}{2} + \sum_{n=1}^{N} [a_n(f)\cos nx + b_n(f)\sin nx] = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s) D_N(s-x)ds,$$

where D_N is as defined in part (a).

(c) Prove that

$$\lim_{N\to\infty} \int_0^{\pi} [\frac{1}{x} - \frac{1}{2\sin(x/2)}] \sin[(2N+1)x/2]dx = 0.$$

(d) Suppose that a function f is piecewise continuous on $[0, \pi]$ and that the right-hand derivative $f_R'(0) = \lim_{h\to 0+} \frac{f(h)-f(0)}{h}$ exists. Prove that

$$\lim_{N\to\infty} \int_0^{\pi} f(s) D_N(s)ds = \frac{\pi}{2} f(0+) = \frac{\pi}{2} \lim_{t\to 0+} f(t).$$

2. Suppose that f is continuously differentiable on $[0, \pi]$ with $f(0) = f(\pi) = 0$. Prove that

$$\int_0^{\pi} |f(x)|^2 dx \le \int_0^{\pi} |f'(x)|^2 dx.$$

[Hint: extend f to be an odd periodic function on $[-\pi, \pi]$, and then do some Fourier analysis.]

3. Give complete solutions to problems 2.7.5 (p. 123), 2.9.3 all parts (p. 131), 3.2.9 (p. 175), 5.1.7 (p. 265), 5.2.8 (p. 268), 5.7.4 (p. 295), 6.1.6 (p. 302-303) and 6.2.5 (p. 311) of the Stade textbook. Recall that Exercise 2.9.3 (a) was assigned as part of HW 6.