University of Colorado

Department of Mathematics

2008/2009 Semester 2 | Math 6360 Complex Variables II | Assignment 1

Selected Solutions

#3 Consider for $r > 0$ $U(r,\theta) = u(r\cos\theta, r\sin\theta)$. For $r > 0$, $u_{xx} + u_{yy} = 0$ if and only if $r^2(u_{xx} + u_{yy}) = 0$. Now viewing U as functions of x and y with $x = r\cos\theta$ and $y = r\sin\theta$ we obtain

$$\frac{\partial U}{\partial r} = \frac{\partial U}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial U}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta.$$

Similarly,

$$\frac{\partial U}{\partial \theta} = \frac{\partial U}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial U}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= -r\frac{\partial u}{\partial x}\sin\theta + r\frac{\partial u}{\partial y}\cos\theta.$$

$$\frac{\partial^2 U}{\partial r^2} = \frac{\partial}{\partial r}[\frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta]$$

$$= \frac{\partial^2 u}{\partial x^2}\cos\theta\cdot\cos\theta + \frac{\partial^2 u}{\partial y\partial x}\cdot\sin\theta\cdot\cos\theta$$

$$+ \frac{\partial^2 u}{\partial x\partial y}\cdot\cos\theta\cdot\sin\theta + \frac{\partial^2 u}{\partial y^2}\cdot\sin\theta\cdot\sin\theta$$

$$= \frac{\partial^2 u}{\partial x^2}\cdot\cos^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}\cdot\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2}\cdot\sin^2\theta.$$

Finally

$$\frac{\partial^2 U}{\partial \theta^2} = \frac{\partial}{\partial\theta}[\frac{\partial U}{\partial\theta}]$$

$$= \frac{\partial}{\partial\theta}[-r\frac{\partial u}{\partial x}\sin\theta + r\frac{\partial u}{\partial y}\cos\theta]$$

$$= -r\frac{\partial^2 u}{\partial x^2}\cdot(-r\sin\theta)\cdot\sin\theta + -r\frac{\partial^2 u}{\partial y\partial x}\cdot(r\cos\theta)\cdot\sin\theta + (-r\frac{\partial u}{\partial x})\cdot\cos\theta$$

$$+ r\frac{\partial^2 u}{\partial x\partial y}\cdot(-r\sin\theta)\cdot\cos\theta + r\frac{\partial^2 u}{\partial y^2}\cdot(r\cos\theta)\cdot\cos\theta - r\frac{\partial u}{\partial y}\sin\theta$$

$$= (r^2)\cdot(\frac{\partial^2 u}{\partial x^2})\cdot\sin^2\theta + r^2(\frac{\partial^2 u}{\partial y^2})\cdot\sin^2\theta + (-2r^2)\frac{\partial^2 u}{\partial x\partial y}\cdot[\sin\theta\cos\theta] - r\frac{\partial u}{\partial x}\cdot\cos\theta - r\frac{\partial u}{\partial y}\sin\theta.$$

From the above we deduce that

$$r\frac{\partial}{\partial r}(r\cdot\frac{\partial U}{\partial r})+\frac{\partial^2 U}{\partial\theta^2}$$

$$= r\cdot(1)\cdot\frac{\partial U}{\partial r}+\ r\cdot r\cdot\frac{\partial^2 U}{\partial r^2}+\frac{\partial^2 U}{\partial\theta^2}$$

$$= r^2\frac{\partial^2 U}{\partial r^2}+r\frac{\partial U}{\partial r}+\frac{\partial^2 U}{\partial\theta^2}$$

$$= r^2\cdot[\frac{\partial^2 u}{\partial x^2}\cdot\cos^2\theta+2\frac{\partial^2 u}{\partial x\partial y}\cdot\sin\theta\cos\theta+\frac{\partial^2 u}{\partial y^2}\cdot\sin^2\theta]$$

$$+\,r\cdot[\frac{\partial u}{\partial x}\cdot\cos\theta+\frac{\partial u}{\partial y}\cdot\sin\theta]$$

$$+\,[(r^2)\cdot(\frac{\partial^2 u}{\partial x^2})\cdot\sin^2\theta+r^2(\frac{\partial^2 u}{\partial y^2})\cdot\sin^2\theta+(-2r^2)\frac{\partial^2 u}{\partial x\partial y}\cdot[\sin\theta\cos\theta]\ -r\frac{\partial u}{\partial x}\cdot\cos\theta-r\frac{\partial u}{\partial y}\sin\theta].$$

$$= r^2\cdot\frac{\partial^2 u}{\partial x^2}\cdot[\cos^2\theta+\sin^2\theta]+r^2\cdot\frac{\partial^2 u}{\partial y^2}\cdot[\cos^2\theta+\sin^2\theta]$$

$$= r^2[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}],$$

with all other terms cancelling out.
So for $r\neq 0$, we have

$$r\frac{\partial}{\partial r}(r\cdot\frac{\partial U}{\partial r})+\frac{\partial^2 U}{\partial\theta^2}=0$$

if and only if

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,$$

as we desired to show.

Prob. 2, p. 171 Define the map P_U from the closed upper-half plane to $\mathbb{R}$ by:

$$P_U(x,y)\ =\ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-\xi)^2+y^2}U(\xi)d\xi,$$

for $y>0$, and $P_U(x,0)=U(x)\ x\in\mathbb{R}$. Recall that P_U is harmonic on the open upper-half plane and continuous on the real axis, by Problem 1. We first note that P_U is bounded since, letting $M=\sup|U(x,y)|$, we know $|P_U(x,0)|=|U(x)|\leq M$, and for $y>0$,

$$|P_U(x,y)|\ \leq\ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-\xi)^2+y^2}\cdot Md\xi$$

$$=\ \frac{M}{\pi y}\int_{-\infty}^{\infty}\frac{1}{(\frac{\xi-x}{y})^2+1}d\xi$$

$$= \text{(substituting } t = \frac{\xi - x}{y}) \ \frac{M}{\pi} \cdot \int_{-\infty}^{\infty} \frac{1}{t^2+1} dt$$
$$= \frac{M}{\pi} \cdot [\arctan t]_{-\infty}^{\infty} \ = \ \frac{M}{\pi} \cdot [\frac{\pi}{2} - (-\frac{\pi}{2}] \ = \ M.$$

We now suppose that u is another function defined on the closed upper-half plane that is harmonic on the open upper-half plane and continuous on the real axis, with $u(x,0) = U(x)$, for all $x \in \mathbb{R}$. Suppose for simplicity that $|u(x,y)| \leq M$ for all (x,y) in the upper-half plane.

Method One: Following the hint, we fix $\epsilon > 0$ and consider the function that is harmonic on the open upper-half plane and continuous on the real axis defined by

$$u(x,y) - P_U(x,y) - \epsilon \ \cdot \mathrm{Im}(\sqrt{iz}),$$

which defines the function everywhere except for $(0,0)$, where we set the function equal to 0. We note that for $z = x + iy$ in the closed upper half plane $\setminus (0,0)$, $\mathrm{Arg}(\sqrt{iz}) \in (\frac{\pi}{4}, \frac{3\pi}{4})$, so that $|z|\frac{\sqrt{2}}{2} \leq \mathrm{Im}(\sqrt{iz}) \ \leq \ |z|$. Hence

$$-|z| \ \leq -\mathrm{Im}(\sqrt{iz}) \ \leq \ -|z|\frac{\sqrt{2}}{2}.$$

It follows that if we fix $R > 0$ and and consider all points $x + iy = re^{i\theta}$, with $0 \leq \ r \leq R$ and $\theta \in [0, \pi]$, we obtain, by using the fact that any harmonic function attains its maximum and minimum on the boundary:

$$h(x,y) = u(x,y) - P_U(x,y) - \epsilon \ \cdot \mathrm{Im}(\sqrt{iz}) \leq \ 2M - \epsilon\frac{R \cdot \sqrt{2}}{2} \ < \ 0$$

if we take R large enough, (i.e. $R > \frac{4M}{\sqrt{2}}$) for $\{(x+iy) = re^{i\theta}, \ 0 \leq r \leq R, \ \theta \in [0,\pi]\}$. (Note that on the interval $[-R, R]$, $h(x,0) = 0$.) It follows that letting ϵ go to 0 we get $u(x,y) - P_U(x,y) \leq 0$, for all points in $\{(x+iy) = re^{i\theta}, \ 0 \leq r \leq R, \ \theta \in [0,\pi]\}$. But we can make R arbitrarily large and get $u(x,y) - P_U(x,y) \leq 0$ for all points in the upper half-plane. A similar argument shows that $P_U(x,y) - u(x,y) \leq 0$ for all points in the upper half-plane. Thus $P_U(x,y) = u(x,y)$ for all points in the upper half-plane.

Method Two: Suppose that u is another harmonic function defined on the upper half-plane that is bounded and agrees with P_U on the real axis. Define $h(x,y) = u(x,y) - P_U(x,y)$, and note that h is bounded, harmonic on the upper half-plane, and continuous on the real axis with $h(x,0) \equiv 0$ there. By the Schwarz reflection principal we can extend h to the lower half-plane by defining $h(x,y) = -h(x,-y)$ for $y < 0$. By Theorem 24 p. 172, h is harmonic (and bounded!) in the entire complex plane. Moreover we can extend h to an entire function f with imaginary part equal to h. By a standard Liouville Theorem type argument, f must be a constant function. This means that h is a constant function as well, and since h is identically 0 on the real axis, we have $h \equiv 0$ throughout $\mathbb{C}$. Hence $u(x,y) = P_U(x,y)$ for all $x+iy$ in the upper half-plane.

Method 3: Extend u to an analytic function on the open upper half-plane, $f = u+iv$. For any z in the open upper half plane, find $\epsilon > 0$ with $\mathrm{Im}(z) > \epsilon$ and $R > 0$ with $R > |z|$. Consider the contour $\mathcal{C}_{\epsilon,R}$ oriented in the counterclockwise direction defined by the line segment $\{x + i\epsilon : -R \le x \le R\}$ and the part of the upper semicircle $\{Re^{it}it : 0 < t < \pi\}$. By Cauchy's Theorem,

$$f(z) = \frac{1}{2\pi i}\int_{\mathcal{C}_{\epsilon,R}} \frac{f(\xi)}{\xi - z} d\xi.$$

Now let $\epsilon \to 0+$ and $R \to \infty$, and the integral along the line seqments tend to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi) + iv(\xi, 0) d\xi$$

and the integral along the circular part goes to 0. Identifying reall parts of both sides we get

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi) d\xi.$$