University of Colorado

Department of Mathematics

2008/2009 Semester 2 Math 6360 Complex Variables II Assignment 2

Selected Solutions

1 p. 178 Choosing the principal branch of the logarithm, we have, for $|\frac{z}{n}| < 1$,

$$\log(1+\frac{z}{n}) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{kn^k},$$

so that

$$\begin{aligned} n\cdot\log(1+\frac{z}{n}) &= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{kn^{k-1}} \\ &= z-\frac{z^2}{2n}+\frac{z^3}{3n^2}-\frac{z^4}{4n^3}+\cdots . \end{aligned}$$

Now for fixed $\rho > 0$, if we take $|z| \le \rho < n$, we have by the summation variant of the dominated convergence theorem, since $\sum_{k=1}^{\infty}|\frac{z^k}{kn^{k-1}}|$ converges absolutely for $|z| \le \rho < n$ and

$$|\sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{kn^{k-1}}| \le \sum_{k=1}^{\infty}|\frac{\rho^k}{kn^{k-1}}|,$$

$$\begin{aligned} \lim_{n\to\infty} n\log(1+\frac{z}{n}) &= \lim_{n\to\infty}[\sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{kn^{k-1}}] \\ &= \sum_{k=1}^{\infty}[\lim_{n\to\infty}(-1)^{k+1}\frac{z^k}{kn^{k-1}}] = z. \end{aligned}$$

It follows that

$$\begin{aligned} \lim_{n\to\infty}(1+\frac{z}{n})^n &= \lim_{n\to\infty} e^{n\cdot\log(1+\frac{z}{n})} \\ &= e^z, \end{aligned}$$

as desired.

#2, p. 178 We have shown in lectures that for any $s_0 > 1$, the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

converges absolutely and uniformly on any half-plane of the form

$$H_{s_0} = \{s\in\mathbb{C} : \mathrm{Re}(s) \ge s_0\}.$$

Since for each fixed $n \in \mathbb{N}$, $\frac{1}{n^s} = e^{-\ln ns}$ is analytic in the entire plane, it follows from the general theory that ζ is infinitely differentiable in its region of convergence which is

$$H = \{s \in \mathbb{C} : \text{Re}(s) > 1\}.$$

Moreover, we may take the derivative in a term-by-term fashion, so that

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^s} = -\sum_{n=2}^{\infty} \frac{\ln n}{n^s}, \; s \in H.$$

1 p. 184 Note that using partial fractions we can write

$$\frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i};$$

solving for A and B we get $A = \frac{i}{2}$ and $B = \frac{-i}{2}$. Hence

$$\frac{1}{z^2+1} = \frac{i}{2(z+i)} - \frac{i}{2(z-i)}$$

$$= \frac{i}{2[(z-a)-(a-i)]} - \frac{i}{2[(z-a)-(i+a))}$$

$$= -\frac{i}{2(a-i)} \cdot \frac{1}{[1-(z-a)/(a-i)]} + \frac{i}{2(a+i)} \cdot \frac{1}{[1-(z-a)/(a+i)]}$$

$$= -\frac{i}{2(a-i)} \sum_{n=0}^{\infty} \frac{(z-a)^n}{(a-i)^n} + \frac{i}{2(a+i)} \sum_{n=0}^{\infty} \frac{(z-a)^n}{(a+i)^n}.$$

This series converges for $|z-a| < |a-i| = |a+i| = \sqrt{a^2+a}$ since a is real. Simplifying, our series becomes:

$$\sum_{n=0}^{\infty} [\frac{i}{2(a+i)^{n+1}} - \frac{i}{2(a-i)^{n+1}}](z-a)^n$$

$$= \sum_{n=0}^{\infty} \frac{i[(a-i)^{(n+1)} - (a+i)^{(n+1)}]}{2(a^2+1)^{(n+1)}}(z-a)^n$$

We now write $a+i = re^{i\theta}$ for $r = \sqrt{a^2+1}$ and $\theta = \arctan \frac{1}{a}$. Then $a - i = re^{-i\theta}$ so that

$$i \cdot [(a-i)^{(n+1)} - (a+i)^{(n+1)}]; = i \cdot [r^{n+1}e^{-i(n+1)\theta} - r^{n+1}e^{i(n+1)\theta}]$$

$$= i \cdot (-2)ir^{n+1} \sin{(n+1)\theta} = 2r^{n+1} \sin{(n+1)\theta}.$$

It follows that the power series expansion is given by

$$\sum_{n=0}^{\infty} \frac{r^{n+1} \sin{(n+1)\theta}}{(a^2+1)^{(n+1)}}(z-a)^n.$$

In the case where $a = 1$ we have $1+i = \sqrt{2}e^{i(\pi/4)}$ so that the power series expansion becomes

$$\sum_{n=0}^{\infty} \frac{\sin{(n+1)(\pi/4)}}{(\sqrt{2})^{(n+1)}}(z-1)^n.$$

We now note that for integers k and $j = 0,\ 1,\ 2,\ 3$ we have $\sin{(4k+j+1)\cdot(\pi/4)} = \sin{[k\pi + (j+1)\pi/4]} = (-1)^k\frac{\sqrt{2}}{2},\ j = 0$, and $\sin{[k\pi + (j+1)\pi/4]} = (-1)^k,\ j = 1$, and $\sin{[k\pi + (j+1)\pi/4]} = (-1)^k\frac{\sqrt{2}}{2},\ j = 2$, and finally $\sin{[k\pi + (j+1)\pi/4]} = 0,\ j = 3$. So in this case our power series becomes

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2}^{4k+2}}(z-1)^{4k} + \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2}^{4k+2}}(z-1)^{4k+1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2}^{4k+4}}(z-1)^{4k+2}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{(2k+1)}}(z-1)^{4k} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{(2k+1)}}(z-1)^{4k+1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{(2k+2)}}(z-1)^{4k+2}$$

#1, p. 190 Recall that the Laurent series expansion for $\pi\cot\pi z$ is given by:

$$\pi\cot\pi z \ = \frac{1}{z} - \frac{\pi^2 z}{3} - \frac{\pi^4 z^3}{45} - \frac{2\pi^6 z^5}{945} - \cdots - \frac{2^{2n}\pi^{2n}B_n}{(2n)!}z^{2n-1} + \cdots,\ |z| < 1.$$

One can obtain this formula by doing a "long division" of the power series expansion for $\sin\pi z$ into $\cos\pi z$.

(i) We showed $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ in class.

(ii) Let $f(z) = \frac{1}{z^4}$. This function has a pole only at $z = 0$. From our discussion on infinite series, we see that

$$\lim_{N\to\infty}[\sum_{n=-N}^{-1} f(n) + \sum_{n=1}^{N} f(n)] \ = \ -\text{Residue at } 0 \text{ for } \frac{\pi\cot\pi z}{z^4} = \frac{\pi^4}{45}.$$

Since f is even we get

$$\lim_{N\to\infty} 2\sum_{n=1}^{N} f(n) \ = \ \frac{\pi^4}{45}.$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} \ = \ \frac{\pi^4}{90}.$$

(iii) Let $f(z) = \frac{1}{z^6}$. This function has a pole only at $z = 0$. From our discussion on infinite series, we see that

$$\lim_{N\to\infty}[\sum_{n=-N}^{-1} f(n) + \sum_{n=1}^{N} f(n)] \ = \ -\text{Residue at } 0 \text{ for } \frac{\pi\cot\pi z}{z^6} = \frac{2\pi^6}{945}.$$

Since f is even we get

$$\lim_{N\to\infty} 2\sum_{n=1}^{N} f(n) \;=\; \frac{2\pi^6}{945}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} \;=\; \frac{\pi^6}{945}.$$