University of Colorado
Department of Mathematics

2008/2009 Semester 2 | Math 6360 Complex Variables II | Assignment 1

Selected Solutions

#1, p. 200 Prove Gauss' formula

$$(2\pi)^{(n-1)/2}\Gamma(z) \;=\; n^{z-\frac{1}{2}}\Gamma(\frac{z}{n})\Gamma(\frac{z+1}{n})\cdots\Gamma(\frac{z+n-1}{n}),\; n\in\;\mathbb{N}.$$

I benefited from Emil Artin's pamphlet "The Gamma Function" and Whittaker and Watson's treatise "A Course of Modern Analysis" in what follows.

Consider the function

$$f(x) \;=\; n^{x-\frac{1}{2}}\Gamma(\frac{x}{n})\Gamma(\frac{x+1}{n})\cdots\Gamma(\frac{x+n-1}{n})$$

for $x > 0$. We have shown in lectures that $\Gamma(x)$ is logarithmically convex, and a small modification of that result shows that $\Gamma(\frac{x+j}{n})$ is log convex for all $j \in \mathbb{N}$. Also one checks directly that $n^{x-\frac{1}{2}}$ is log convex since its second derivative (in x) is zero. Thus f is log convex. We calculate:

$$f(x+1) \;=\; n\cdot n^{x-\frac{1}{2}}\Gamma(\frac{x+1}{n})\Gamma(\frac{x+2}{n})\cdots\Gamma(\frac{x+n-1}{n})\Gamma(\frac{x+n}{n})$$

$$=\; n\cdot n^{x-\frac{1}{2}}\Gamma(\frac{x+1}{n})\Gamma(\frac{x+2}{n})\cdots\Gamma(\frac{x+n-1}{n})\Gamma(\frac{x+n}{n})\Gamma(\frac{x}{n})\cdot\frac{x}{n} \;=\; x\cdot f(x).$$

Thus the function f satisfies some of the conditions of the Bohr-Mohlerupp Theorem, and there must exist a positive constant c_n such that

$$f(x) \;=\; c_n\cdot\Gamma(x),\; x>0.$$

Our aim is to show that this positive constant is $(2\pi)^{(n-1)/2}$. We can then extend the equality to the complex plane minus the poles of Γ, in the usual fashion. This amounts to showing that

$$n^{\frac{1}{2}}\cdot\Gamma(\frac{1}{n})\Gamma(\frac{1+1}{n})\cdots\Gamma(\frac{n-1}{n}) \;=\; (2\pi)^{(n-1)/2}.$$

We now consider

$$[n^{\frac{1}{2}}\cdot\Gamma(\frac{1}{n})\Gamma(\frac{1+1}{n})\cdots\Gamma(\frac{n-1}{n})]^2 \;=\; n\cdot\prod_{j=1}^{n-1}[\Gamma(\frac{j}{n})\Gamma(\frac{n-j}{n})]$$

$$= n^{-1} \prod_{j=1}^{n-1} \frac{\pi}{\sin(j\pi/n)} = \frac{n(2\pi)^{n-1}}{n} = (2\pi)^{(n-1)}.$$

Now we have $\Gamma(\frac{j}{p}) > 0$ for $1 \leq j \leq p-1$ so that taking square roots we obtain

$$n^{\frac{1}{2}} \cdot \Gamma(\frac{1}{n})\Gamma(\frac{2}{n}) \cdots \Gamma(\frac{n-1}{n}) = (2\pi)^{(n-1)/2},$$

as desired, so that

$$n^{x-\frac{1}{2}}\Gamma(\frac{x}{n})\Gamma(\frac{x+1}{n}) \cdots \Gamma(\frac{x+n-1}{n}) = (2\pi)^{(n-1)/2}\Gamma(x), \; x > 0.$$

By the theory of meromorphic functions we get

$$n^{z-\frac{1}{2}}\Gamma(\frac{z}{n})\Gamma(\frac{z+1}{n}) \cdots \Gamma(\frac{z+n-1}{n}) = (2\pi)^{(n-1)/2}\Gamma(z),$$

for all $z \in \mathbb{C}$ where both sides are defined.
Another way to do the problem is to note that as in the proof of the Legendre duplication formula,

$$\frac{d}{dz}[\sum_{j=0}^{n-1}(\frac{\Gamma'(z+\frac{j}{n})}{\Gamma(z+\frac{j}{n})})] = n\frac{d}{dz}(\frac{\Gamma'(nz)}{\Gamma(nz)}).$$

Integrate twice to get

$$\prod_{j=0}^{n-1}\Gamma(z+\frac{j}{n}) = e^{cz+d}\Gamma(nz)$$

for some constants c and d, so that

$$\prod_{j=0}^{n-1}\Gamma(\frac{z+j}{n}) = e^{c\frac{z}{n}+d}\Gamma(z).$$

First set $z = \frac{1}{n}$, then set $z = 1$ to get $c = -n\ln n$ and $d = \frac{n-1}{2}\ln 2\pi + \frac{1}{2}\ln n$. You still need to know that

$$\Gamma(\frac{1}{n})\Gamma(\frac{2}{n}) \cdots \Gamma(\frac{n-1}{n}) = (2\pi)^{\frac{n-1}{2}} n^{-\frac{1}{2}}$$

to use this method.