University of Colorado
Department of Mathematics

2008/2009 Semester 2 | Math 6360 Complex Variables II | Assignment 4

Selected Solutions

Q #4 Recall that

$$\sin z \;=\; \sin x \cosh y + i \cos x \sinh y,$$

so that if $w = u(z) + iv(z) = \sin z$, then $u(z) = \sin x \cosh y$ and $v(z) = \cos x \sinh y$. Let $B = \frac{\pi}{2},\ A = \frac{\pi}{2} + iy,\ 0 \leq\ y\ \gamma$. In the segment $\overline{BA}$, then the map $\sin z$ takes $(\frac{\pi}{2}+iy)$ to $\cosh y + i0 = \cosh y$, since $\cos\frac{\pi}{2} = 0$, so that a point $(\frac{\pi}{2}, y)$ in the z-plane is mapped to $(\cosh y, 0)$ in the w-plane, so that the vertical ray from from B to $\frac{\pi}{2}+i\infty$ is mapped to the horizontal ray $[1,\infty)$. Similarly, if $C = -\frac{\pi}{2},\ D = -\frac{\pi}{2}+iy,\ 0 \leq\ y\ \gamma$, then a point $(-\frac{\pi}{2}, y)$ in the z-plane is mapped to $(-\cosh y, 0)$ in the w-plane. Thus in this case, the vertical ray from from C to $-\frac{\pi}{2} + i\infty$ is mapped to the horizontal ray $(-\infty, -1]$. If we consider the horizontal line segment from $C = -\frac{\pi}{2}$ to $B = \frac{\pi}{2}$, since $\cosh 0 = 1$ and $\sinh 0 = 0$, the point x for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ in the z-plane will map to $\sin x$ in the w-plane, and since $\sin x$ takes on all values from -1 to 1 as x ranges over the interval in question, the image of the segment $\overline{CB}$ is the horizontal segment $[-1, 1]$. So the boundary of our region in the z-plane is mapped onto the boundary of the upper-half plane in the w-plane, i.e. is mapped onto the real line in the w-plane. We now just want to show that the interior of our half-strip is mapped one-to-one and onto the upper half plane. We will do this by considering the image of vertical rays in the interior of our strip. If $0 < c < \frac{\pi}{2}$, the points on the vertical line $x = c$ in the z-plane are mapped to points on the curve

$$u = \sin(c)\cosh y,\ v = \cos(c)\sinh y,\ -\infty\ < y\ < \infty,$$

which is the right-hand branch of the hyperbola $\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1$, (since $\cosh^2 y - \sinh^2 y = 1$ for all $y \in \mathbb{R}$, and $\sin(c) > 0$). Note for $y \geq 0$, we have $\sinh y \geq 0$ and hence $v \geq 0$, so that in fact, we only obtain that part of the right-hand branch of the hyperbola that lies above the x-axis, i.e. in the upper half-plane. Similarly, if $-\frac{\pi}{2} < c < 0$, the points on the vertical line $x = c$ in the z-plane are mapped to points on the curve

$$u = \sin(c)\cosh y,\ v = \cos(c)\sinh y,\ -\infty\ < y\ < \infty,$$

which is the left-hand branch of the hyperbola $\frac{u^2}{\sin^2(c)} - \frac{v^2}{\cos^2(c)} = 1$, (again, since $\cosh^2 y - \sinh^2 y = 1$ for all $y \in \mathbb{R}$, and since $\sin(c) < 0$). Again, for $y \geq 0$, we have $\sinh y \geq 0$ and hence $v \geq 0$, as we'll have $\cos(c) > 0$, so that once more we only obtain that part of the right-hand branch of the hyperbola that lies in the upper half-plane.

We need to consider the vertical ray $x = 0,\ y \geq 0$ separately. By inspecting the formulas for u and v, we see that the image of a point $(0, y)$ is $(0, \sinh y)$, and as y goes from 0 to $+\infty$, the values of $\sinh y = \frac{e^y - e^{-y}}{2}$ go from 0 to $+\infty$ too. Thus the vertical ray $x = 0,\ y \geq 0$ is mapped one-to-one and onto the vertical ray $u = 0,\ v \geq 0$ in the w-plane by our mapping.

We finally note that each point in our domain strip is contained on exactly on of the vertical rays $x = c,\ y \geq 0$, and for $c \neq 0$, a corresponding point on this line is mapped one to one onto the upper half of a hyperbola branch. We note that as c gets closer to $+\frac{\pi}{2}$, the branch of the hyperbola that $(c, y),\ y \geq 0$ is mapped onto bends down closer and closer to the segment $[1, \infty)$ of the real line. Similarly, as c gets closer and closer to $-\frac{\pi}{2}$, the branch of the hyperbola that $(c, y),\ y \geq 0$ is mapped onto bends down closer and closer to the segment $(-\infty, 1]$ of the real line. Thus the image of each point in the interior of the strip lies in the upper half-plane $v > 0$ of the w-plane, and moreover, each element in the upper half-plane is the image of precisely one element of the interior of the strip.

Thus the map $w = \sin(z)$ is a one-to-one map of the strip in question onto the closed upper half-plane $v \geq 0$.

Q #5(b) Using the transformation proved in part (a),

$$w(z) = e^{\frac{3\pi i}{4}} \int_0^z (s+1)^{-3/4} s^{-1/2} (s-1)^{-3/4} ds,$$

one calculates that w maps the upper half plane onto an isosceles triangle with vertices

$$w(0) = e^{\frac{3\pi i}{4}} \int_0^0 (s+1)^{-3/4} s^{-1/2} (s-1)^{-3/4} ds = 0,$$

$$\begin{aligned} w(1) &= e^{\frac{3\pi i}{4}} \int_0^1 \ s^{-1/2}(s^2-1)^{-3/4} ds \\ &= e^{\frac{3\pi i}{4}} \int_0^1 \ s^{-1/2}(1-s^2)^{-3/4}(-1)^{-3/4} ds \\ &= e^{\frac{3\pi i}{4}} e^{-\frac{3\pi i}{4}} \int_0^1 \ s^{-1/2}(1-s^2)^{-3/4} ds = b, \end{aligned}$$

for $b = \int_0^1 (1-s^2)^{-3/4} s^{-1/2} ds$. Finally,

$$w(-1) = \ e^{\frac{3\pi i}{4}} \int_0^{-1} \ s^{-1/2}(1-s^2)^{-3/4}(-1)^{-3/4} ds$$

which, upon making the subsitution $s = -t$, becomes

$$e^{\frac{3\pi i}{4}} \int_0^1 (-t)^{-1/2}(1-t^2)^{-3/4}(-1)^{-3/4}(-1) dt$$

$$= e^{\frac{3\pi i}{4}} e^{-\frac{3\pi i}{4}} e^{-\frac{1\pi i}{2}} (-1) \cdot \int_0^1 t^{-1/2}(1-t^2)^{-3/4} dt = bi,$$

where b is the definite integral defined above (which also can be defined in terms of beta functions.

The Schwarz-Christoffel theory shows that the real line maps onto the boundary of the triangle and the open upper half-plane maps to the interior of the triangle.