University of Colorado
Department of Mathematics

Semester 1 (2009/10) **Math 8330 Functional Analysis 1** **Assignment 2**

Due Wednesday September 30, 2009

1. Do problems p. 33: 2.6 (b), (c), (d), (e); p. 47: 3.2 (g); p. 48: 3.4 (a),(b),(c) in the Baggett textbook.

2. Let E be a normed vector space, with norm $\|\cdot\|$, and let $\mathcal{T}_{\|\cdot\|}$. be the topology on E induced by the metric coming from its norm. On the other hand, let $\mathcal{W}_{\|\cdot\|}$ be the weak topology on E induced by the functions $\{\|\cdot\ -y\| : E \to [0,\infty) : y \in E\}$. Show that these two topologies are the same.

3. (a) Let E be a topological vector space, let U be a non-empty convex open neighborhood of $\vec{0}$ in E, and let ρ_U be the subadditive functional called the Minkowski gauge defined on E by

$$\rho_U(x) = inf\ \{r > 0 : x \in rU\}$$

(see pp. 55-56 of the text). Prove that if U is in addition balanced, i.e. if $\lambda \cdot x \in U$ whenever $x \in U$ and $|\lambda| \leq 1$, then ρ_U is a seminorm on E.

(b) Suppose that E is a vector space over the field $\mathbb{R}$ or over $\mathbb{C}$, and that $p : E \to [0,\infty)$ be a seminorm on E. Let $\mathcal{W}_p$ be the weak topology on E induced by the functions $\{p_y : E \to [0,\infty)\}$, where $p_y(x) = p(x-y)$. Prove that $(E,\ \mathcal{W}_p)$ is a topological vector space if and only if p is a norm. (Hint: prove that $(E,\ \mathcal{W}_p)$ is T_0 if and only if p is a norm).

4. (a) Let $\mathcal{S}$ be the Schwarz space defined in lectures. Prove that $x^k e^{-x^2} \in \mathcal{S}$ for every integer $k \geq 0$.

(b) Recall that for each $n \in \mathbb{N} \cup \{0\}$ there is norm p_n defined on $\mathcal{S}$ by

$$p_n(f) = sup_{x\in\mathbb{R}}\ max_{0\leq i,j\leq n}\ |x^j\ f^{(i)}(x)|.$$

Show that with respect to the topology $\mathcal{S}$ induced by the seminorms p_n, the differentiation map $\mathcal{D} : \mathcal{S} \to \mathcal{S}$ is continuous.