

University of Colorado
Department of Mathematics

Semester 1 (2009/10)

Math 8330 Functional Analysis 1

Assignment 3

Due Monday, October 19, 2009

1. Do problems p. 52: 3.8 (a), (b), (c); p. 54: 3.11 (d), (e); p. 56: 3.12, in the Baggett textbook.
2. Let U be a non-empty open subset of \mathbb{C} and define $H(U)$ to be the set of all complex-valued **analytic** functions defined on U . Prove that $H(U)$ is a complex vector space. For any compact subset $K \subseteq U$, define a seminorm p_K on U by

$$p_K(f) = \sup \{|f(z)| : z \in K\}.$$

Let $H(U)$ be given the topology determined by the seminorms $\{p_K\}$. Prove that this topology is equivalent to the topology of uniform convergence on compact sets, and that with respect to this topology, $H(U)$ is complete, i.e. every Cauchy net in $H(U)$ converges to an element of $H(U)$.

3. (i) Let X be a topological vector space over the field \mathbb{R} or \mathbb{C} , and let L be a proper subspace of X . Show that L contains no non-empty open set U of X .

(ii) Let X be a topological vector space over the field \mathbb{R} or \mathbb{C} . Suppose that A is a convex subset of X , and $\alpha, \beta > 0$. Prove that $\alpha A + \beta A = (\alpha + \beta) A$. Prove that this statement need not be true if A is not convex.
4. Let X be the real Hilbert space $L^2[0, 1]$. For $\alpha \in \mathbb{R}$, let X_α be the subset of X consisting of all continuous real-valued functions $f : [0, 1] \rightarrow \mathbb{R}$ such that $f(1/2) = \alpha$. Prove that each X_α is convex, and each is norm dense in X . Deduce that for $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta$, the sets X_α and X_β are disjoint convex sets which cannot be separated by any continuous linear functional on X .