University of Colorado
Department of Mathematics

Semester 1 (2009/10) **Math 8330 Functional Analysis 1** **Assignment 4**

Due Friday November 6, 2009

1. Do Exercise 4.2 all, pp. 67-68, Exercise 4.3, (b), p. 69, Exercise 5.1 (b), (c), (d) , p. 81, and Exercise 5.9 (a), (b), p. 90, in the Baggett textbook.
2. (a) Let X be a topological vector space, and suppose that A and B are two subsets of X which are bounded, in the sense defined in lectures. Prove that $A+B$ is bounded.

   (b) Let X be a topological vector space over the field $\mathbb{R}$ or $\mathbb{C}$, and let U be a non-empty subset of X. Prove that U is bounded if and only if whenever $\{x_n\}$ is a sequence in U and $\{\alpha_n\}$ is a sequence of scalars such that $\alpha_n \to 0$ as $n \to \infty$, then $\alpha_n x_n \to 0$ as $n \to \infty$.
3. Let $\overline{B_1}$ be the closed unit ball in $L^p[0,1]$, for $1 < p < \infty$, i.e.

$$\overline{B_1} = \{f \in L^p[0,1] : \|f\|_p = (\int_0^1 |f(x)|^p dx)^{1/p} \leq 1\}.$$

   Prove that the extreme points of $\overline{B_1}$ are exactly the elements of the unit sphere of $\overline{B_1}$, i.e. the elements of $\overline{B_1}$ for which $\|f\|_p = 1$. [Hint: Use the fact that $|(1-t)b + tc|^p < (1-t)|b|^p + t|c|^p$ if $b \neq c$ and $0 < t < 1$.]
4. (a) Show that the space $\mathbf{c}$ of all convergent sequence of complex numbers is a Banach space, where we define $(a_n) + (b_n) = (a_n + b_n) \in \mathbf{c}$, $\alpha \cdot (a_n) = (\alpha a_n)$, $\|(a_n)\| = sup_{n\in\mathbb{N}}\{|a_n|\}, (a_n), (b_n) \in \mathbf{c}$, $\alpha \in \mathbb{C}$.

   (b) Let $\mathbf{c_0}$ be the sequence space over $\mathbb{C}$ consisting of all $\mathbb{C}$-valued sequences whose limit is 0, i.e.$\mathbf{c_0} = \{(\xi_n)_{n=1}^\infty : lim\ \xi_n = 0\}$. Note that $\mathbf{c_0}$ is a subspace of $\mathbf{c}$. Prove that $\mathbf{c_0}$ is a closed subspace, hence a Banach space, and verify that the dual space of $\mathbf{c_0}$ can be identified with the Banach space of absolutely summable sequences $l^1(\mathbb{N})$.