University of Colorado
Department of Mathematics

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1. Problems posed by Chuck Akemann: Let $\mathcal{A}$ be a masa of a von Neumann algebra $\mathcal{N}$, let f_0 be a normal state of $\mathcal{A}$. Let $\mathcal{S}$ equal the set of state extensions of f_0 to $\mathcal{N}$. We know that $\mathcal{S}$ is a weak-* compact convex subset of $\mathcal{N}^*$.
   (a) When is $\mathcal{S}$ a singleton? This is the case if $\mathcal{A} = \mathcal{N}$ of course, but what about the general case?
   (b) When is $\mathcal{S}$ finite dimensional?
   (c) When is $\mathcal{S}$ norm compact?
   (d) When does $\mathcal{S}$ contain a non-normal element?
   (e) When does $\mathcal{S}$ contain a singular element?
   (f) What are the weak-* closed faces of $\mathcal{S}$; especially, what are the extreme points of $\mathcal{S}$?
   (g) Let $\mathcal{S}_{\mathcal{A}}$ denote the set of elements g in $\mathcal{S}$ such that $\mathcal{A}$ is contained in the centralizer of g, i.e. for all a in $\mathcal{A}$ and b in $\mathcal{N}$, $g(ba) = g(ab)$. What are the answers to Questions (a)-(f) for $\mathcal{S}_{\mathcal{A}}$?
   (h) For what pairs $(\mathcal{N}, \mathcal{A})$ can we say that a state f of $\mathcal{N}$ (that is an extension of f_0) is normal iff f_0 is normal?

   As far as Chuck Akemann knows, NONE of these questions is more than partially solved.

2. Problems posed by Dietmar Bisch:
   (a) Given the standard invariant of a non-amenable subfactor $\mathcal{N}$ of the hyperfinite II_1 factor $\mathcal{R}$, is there another subfactor $\mathcal{Q}$ of $\mathcal{R}$ such that the pair $(\mathcal{Q}, \mathcal{R})$ is not isomorphic to the pair $(\mathcal{N}, \mathcal{R})$, but $P_{\mathcal{N}\subset\mathcal{R}}$ is isomorphic to $P_{\mathcal{Q}\subset\mathcal{R}}$? Bisch and Popa have considered related problems.
   (b) Which standard invariants are realized by subfactors of the hyperfinite II_1 factor? Popa and Shylatenko have shown that all invariants can be realized by subfactors of $L(F_\infty)$, but it is not known whether there are standard invariants that cannot arise as subfactors of other types of factors.
   In the discussion, a participant asked a related question to Bisch, and came up with an idea. He says that he and Nica Popa have come up with an idea involving explicit constructions.
   Dietmar Bisch says this particular question is more precise, and is more subtle

than coming up with some subfactors, although the relative fundamental group might be helpful. A result of Jones that is converse to Ocneanu's Theorem. involves the standard invariant. Bisch asks: is there an analog of that theorem for subfactors?

(c) What are planar relations, i.e. how should they be defined? There are many examples of such relations - he gave a notion of the free product, but he wants to do something more concrete, allowing him to construct examples. The free case corresponds to Temperly-Lieb relations. Larger planar algebras throw in more relations, conditional expectations, etc., and technically, things can be hard to control from this point of view. There are relations that are useful - the exchange relation, for example. This is an element in $\mathcal{N}' \cap \mathcal{M}_1$, that is not in the T-L subset $\mathcal{N}' \cap \mathcal{M}_1$.
At this point, Bisch draws a diagram representing pictures of relations, knots etc. first made by Jones and Bisch, gave rise to the construction of Catalan subfactors in this way. There is no general theory of what planar relation should be. In this direction, there has been recent progress, made by Emily Peters, Scott Morrison, Steven Bigelow, Morris Snyder, etc. This has led to new constructions of Haagerup subfactors. He hopes these ideas can be generalized to find new subfactors. Subfactors up to index 5 or $3+\sqrt{3} \sim 4.72$ were done using these people's construction. He hopes the idea can be a powerful tool. Currently there are lots of examples of planar relations. He hopes the general theory would be of concrete type.
Indeed, this problem is very combinatorial, one doesn't need to know anything about subfactors to work on this problem. In particular, it is a nice problem because it will show connections between various areas of mathematics: subfactor theory, quantum mechanics, low dimensional topology - perhaps this is tip of an iceberg!

3. Problems posed by George Elliott:

(a) The main topic he has been thinking about is the classification of C^*-algebras in general. To what extent is it possible to classify separable C^*-algebras? Fifteen years ago at the Field's Institute, he gave a talk: "K-theory suffices " (for arbitrary nuclear separable C^*-algebras - aka amenable C^*-algebras). Since then, people have been trying to talk him down from this hope, but he remains stubborn. What does K-theory mean? It's not just K_0, K_1, he's also thinking of things that are invariant under conjugation by unitaries. E.g. traces are invariants. Throw in circles, instead of taking limits of finite dimensional C^*-algebras, where one just needs K_0, take limits of circle algebras, you need K_1 too. These groups suffice for C^*-algebras of real rank 0, but if the C^*-algebras are not of real rank 0, we need traces, too. Recently, the Cuntz semigroup has been considered – this combines K_0 and traces. Take low-dimensional spaces, as Guihua Gong mentioned in his talk. In the case of simple C^*-algebras, it is enough to consider limits of matrix algebras over spaces of dimension 3. Take simple inductive limits, and use the same invariants. This already involves some work, but if you take direct limit of matrix algebras over metric spaces, (for example, J. Villadsen considered the infinite dimensional torus), you get algebras that you don't know what to do with. They are not of Gong's type. This construction didn't convincingly show that C^*-algebras couldn't be classified by K - groups, since the positive cones were "solid", i.e. no elements left out, it was unperforated. Andrew Toms showed it is possible to construct simple C^*-algebras where the K_0 group the countable abelian group that you want (e.g. the group of rational numbers) but his C^*-algebras are not isomorphic to AF or AT algebras. So there are examples not classified by K_0. Andrew Toms came up with an example with $K_0 = \mathbb{Q}$, and $K_1 = \{0\}$, and Toms also showed it was not isomorphic to the standard example, by means of a K-theoretic invariant (Cuntz's ordered semi-group). So, Elliott suggests, why can't the Cuntz semigroup be included as a K-theoretic invariant? However, on the other hand, no one has shown Cuntz semigroup is sufficient, even in the simple case. So Elliott suggests: prove that a simple AH algebra must be in the Elliott-Gong-Li class, if its Cuntz semigroup says so, + K_1. In this case, ordered K_0, K_1, and traces should suffice for classification.

(b) Ask the same question in the non-simple case, approximately one-dimensional case, and also the AT case where the C^*-algebras that arise are non-simple. (The simple case has already been solved by Qing Lin). If you carry out this procedure with intervals (closed intervals [0,1]) then adding the Cuntz semigroup gives sufficient information for classification. If instead of intervals, you take trees this works too. Ellliot warns if you look closely at his abstract, $K_1 = \{0\}$, these methods still work; however, we don't know what to do in general cases where $K_1 = \{0\}$. Intervals are fine, but circles do not work out fine. Inductive limit of spaces of at most three. Considering one-dimensional spaces, it can be done with circles if the limit C^*-algebra is simple, but in the non-simple case, this is false.

(c) A recent result of Christensen, Sinclair, and Smith, and also White and Winter - - if two concrete C^*-algebras are "close" (represented in the same Hilbert space) with respect to the Hausdorff metric on the unit ball, and are nuclear, then they are isomorphic. It seems that this is a related question: if two nuclear C^*-algebras are "close", that means that all significant invariants you can think of are the same. Eric Christensen showed that if two AF algebras are "close", then their ordered K_0-groups are the same, so isomorphic. So this seems close to stating that enough invariants of simple nature are close and in fact the same. In some cases, it is easy to guess what invariant should be the Cuntz invariant.

(d) Jiang, Su and Elliott have shown that if you take a circle, you can change it into a Cantor set by cutting. If you are considering two circles, you have two unitary elements. In relation to the commutative rotation algebra, if you "cut out" the other circle, you can obtain an AF algebra. Why isn't this true for non-commutative rotation algebras? The coauthors have been able to prove that for almost all rotation algebras (a dense G_δ of them), you can use this procedure to get an AF -algebra. In the sense of Lebesgue measure, the complement of a dense G_δ set complement must have Lebesgue measure 0. But there are some intractable cases in this complement that can't be computed. Elliott aims to look at the remaining cases. Choi, Ruan and Elliott showed that gaps in the spectrum, must exist for a dense G_δ - for a long time he didn't know this, but took for granted that this fact was known. Another example that hasn't yet been done is whether or not this procedure applied to general irrational numbers gives sets other than a Cantor set. Can one obtain single gap in this manner?

4. Problems posed by Kathy Merrill:

   (a) One aim: build a "universal Hilbert space" for GMRA's, i.e. a Hilbert space in which all different equivalence classes of GMRA spaces can be embedded as a subspace. Recall that "canonical" GMRA's can be constructed from specific triples $\{(m, h, g)\}$ where m is a multiplicity function, h is the family of "low-pass" filters, and g is the family of "high-pass" filters.

   (b) In her conference lecture, Merrill demonstrated two inequivalent GMRA's, one being dilation by a dilation matrix, and translation by integers in $L^2(\mathbb{R}^d)$, and the other examples being fractal GMRA's. For example, the Jorgensen-Dutkay inflated Cantor set $\mathcal{R}$ is the unions of dilates of translates of Cantor set. There is a natural idea of nesting of fractal set, leading to a natural MRA structure here: $\mathcal{R} = \cup_{j,\,k} 3^j(C + k)$. Consider the MRA whose scaling function is χ_C, where C is the Cantor set, and let V_0 be the closed span in $L^2(\mathcal{R})$ of integer translates of χ_C. In this case, the "low-pass" filter is $\frac{1}{\sqrt{2}}(1 + e_2(x))$. This is the filter h building the MRA in $L^2(\mathcal{R})$. Not much is know about these. If we are given an given triple $(m,\ h, g)$ and build the associated canonical GMRA from this triple, we don't know if it sits in a Hilbert space corresponding to an inflated fractal space, in a Hilbert space corresponding to $L^2(\mathbb{R}^d)$, or neither.

   (c) The related question is the following: is there even one other filter, which filters can be embedded into Cantor fractal space $\mathcal{R}$? Palle Jorgensen and Dorin Dutkay know that if a filter is too smooth, it won't work. She wants to know about filters related to wavelet sets.