

On the Toeplitz algebra of the affine semigroup over the natural numbers

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GPOTS - Boulder

(joint work with Iain Raeburn)

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Q: What is a quasi-lattice ordered group?

A: It is a pair (G, P) consisting of a group and a unital subsemigroup with no inverses in which the left cones satisfy

$$xP \cap yP = \begin{cases} zP & \text{when } xP \cap yP \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

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Q: Give examples.

(1) (\mathbb{Z}, \mathbb{N}) ; \leq is the usual total order; $x \vee y = \max\{x, y\}$. [1/2 point]

(2) $(\mathbb{Z}^d, \mathbb{N}^d)$; \leq is the usual partial order (not total) on \mathbb{Z}^d ;

$x\mathbb{N}^d \cap y\mathbb{N}^d = (x \vee y)\mathbb{N}^d$, where $(x \vee y)_i = \max\{x_i, y_i\}$. Every pair has a supremum so this is a lattice order. [1 point]

(3) $(\mathbb{Q}_+^*, \mathbb{N}^\times)$; an infinite dimensional version of (2), written multiplicatively, with one generator for each prime number. This one is lattice order too, and $m \vee n = [m, n]$, the l.c.m. of m and n . [1 point]

(4) $(\mathbb{F}_2, \mathbb{F}_2^+)$, the free group on d generators $\{a, b\}$ with its canonical free submonoid generated by the same generators (but not their inverses). This one is not a lattice since $a\mathbb{F}_2^+ \cap b\mathbb{F}_2^+ = \emptyset$. [2 points]

(5) (A_Γ, A_Γ^+) an Artin group of rectangular type associated to the simplicial graph Γ , with its canonical Artin monoid. Some pairs of generators commute, while the remaining pairs are relatively free, according to the connectivity of Γ . [Crisp-L, JAMS 02] This case combines (2) and (4) above, so it gets [3 points].

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Q: Why do we study quasi-lattice ordered groups? (explain your answer)

A: Because their **Toeplitz C*-algebras** $C^*(T_x : x \in P)$ are nice.

Explanation: when P is quasi-lattice ordered,

$$(T_x T_x^*)(T_y T_y^*) = \begin{cases} T_{x \vee y} T_{x \vee y}^* & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty. \end{cases}$$

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A deep insight of Andu Nica was to turn this property into a definition of covariant representations.

Q: What is a covariant representation?

Def: It is a representation $V : P \rightarrow B(H)$ by isometries such that

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Q: () So what?

A: Covariant representations allow Wick-ordering of products:

$$V_x^* V_y = V_{x^{-1}(x \vee y)} V_{y^{-1}(x \vee y)}^*.$$

Hence: $C^*({V_x : x \in P}) = \overline{\text{span}}\{V_x V_y^* : x, y \in P\}$ (this is the meaning of “nice”).

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Q: Give interesting examples of covariant representations

1) (\mathbb{Z}, \mathbb{N}) . Any isometric representation. Covariance is *automatic* for all total orders. [correct but,,, 0 points]

2) $(\mathbb{Z}^2, \mathbb{N}^2)$. Covariance is equivalent to “*-commuting” generators. $ST = TS$ and $S^*T = TS^*$ [1 point] HW: (for extra credit) Find a representation that is NOT covariant. Hint: a proper isometry does not *-commute with itself.

3) $(\mathbb{Q}_+^*, \mathbb{N}^\times)$. Covariance translates into $V_m^*V_n = V_nV_m^*$ if $(m, n) = 1$, which, incidentally, is one of the relations in the presentation of the Bost-Connes C^* -algebra. [1 point]

4) $(\mathbb{F}_2, \mathbb{F}_2^+)$. Covariance means generators have orthogonal ranges: $V_a^*V_b = 0$ for a, b generators of \mathbb{F}_2^+ . [2 points]

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Why are these representations called “covariant”?

A: Hold it, this was NOT covered in lecture! Anyway...

For each $y \in P$ let 1_y denote the characteristic function of the cone $yP = \{z \in P : y \leq z\}$. Then the quasi-lattice condition implies that

$$B_P := \overline{\text{span}}\{1_y : y \in P\} \subset \ell^\infty(P)$$

is a (commutative) C^* -algebra, on which there is a semigroup action of P by injective endomorphisms $\alpha_x(1_y) = 1_{xy}$.

The *covariant isometric representations* of P are in one to one correspondence with the *covariant pairs* for the semigroup dynamical system (B_P, P, α) :

$$V \leftrightarrow (\pi_V, V),$$

a pair (π, V) is covariant if $\pi(\alpha_x(f)) = V_x \pi(f) V_x^*$ for $f \in B_P$.

This realises the “crossed-product-like” structure observed by Nica, and serves as a framework for the analysis of Toeplitz algebras and their quotients:

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The *covariant isometric representations* of P are in one to one correspondence with the *covariant pairs* for the semigroup dynamical system (B_P, P, α) :

$$V \leftrightarrow (\pi_V, V),$$

a pair (π, V) is covariant if $\pi(\alpha_x(f)) = V_x \pi(f) V_x^*$ for $f \in B_P$.

This realises the “crossed-product-like” structure observed by Nica, and serves as a framework for the analysis of Toeplitz algebras and their quotients:

Proposition (L-Raeburn JFA 96)

The universal C^ -algebra for covariant isometric representations of P is canonically isomorphic to the semigroup crossed product $B_P \rtimes_\alpha P$.*

Q: So what?

A: Crossed products have symmetries:

- 1 $B_P \rtimes_\alpha P$ carries a natural (dual) coaction

$$\delta : B_P \rtimes_\alpha P \rightarrow B_P \rtimes_\alpha P \otimes C^*(G),$$

given by

$$\delta(V_x V_y^*) = V_x V_y^* \otimes \epsilon_{xy^{-1}}.$$

- 2 When G is abelian δ may be viewed as a dual action $\hat{\alpha}$ of \hat{G} on $B_P \rtimes_\alpha P$.
- 3 There is a faithful conditional expectation $E : B_P \rtimes_\alpha P \rightarrow B_P$ such

$$\text{that } V_x V_y^* \mapsto \begin{cases} V_x V_x^* & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

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Q: Characterise the Toeplitz C^* -algebra of a qlo.

Theorem (Nica 92, L-Raeburn 96)

If (G, P) has a certain amenability property (which is automatic if G is an amenable group and also holds for many nonamenable G), then the Toeplitz algebra $\mathcal{T}(P)$ is universal for covariant isometric representations.

Explicitly: if

$$\begin{aligned}V_x^* V_x &= 1, \\V_x V_y &= V_{xy} \\V_x V_x^* V_y V_y^* &= \begin{cases} V_{x \vee y} V_{x \vee y}^* & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty, \end{cases}\end{aligned}$$

then the map $T_x \mapsto V_x$ extends to a C^* -algebra homomorphism

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Q: How do you recognise the Toeplitz C^* -algebra of a qlo semigroup when you see it?

A: Assuming V is a covariant representation of P , the map

$$\phi_V : \mathcal{T}(P) \rightarrow C^*({V_x : x \in P})$$

is an *isomorphism* iff the given isometries are “**jointly proper**”, i.e.

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Q: What if the “joint properness” condition fails?

A: There is an extremal (yet not catastrophic) way for this to happen.

Def. A subset $F \subset P \setminus \{e\}$ is **basic** if for each $z \in P \exists f \in F$ with $f \vee x < \infty$. (idea: the cones on F “cover the boundary at ∞ ”)

Def: The *boundary ideal* \mathcal{I}_b of $\mathcal{C}^*(P)$ is the one generated by all products $\prod_{x \in F} (1 - V_x V_x^*) = 0$, with F a finite **basic** subset of P .

Def: The **boundary quotient** is $\mathcal{C}^*(P)/\mathcal{I}_b$; the unital C^* -algebra generated by elements $V_x : x \in P$ subject to the relations

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Theorem (L-Crisp, 07)

Under suitable amenability and topological freeness assumptions on (G, P) , the boundary quotient is purely infinite simple.

Q: What are the boundary quotients of the examples?

- 1 (\mathbb{Z}, \mathbb{N}) . The singleton $\{1\}$ is the smallest basic set and suffices; the boundary condition is $1 - VV^* = 0$, i.e. V is a unitary and thus the boundary quotient is $C^*(\mathbb{Z})$.
- 2 $(\mathbb{Z}^2, \mathbb{N}^2)$. The minimal basic sets are $\{(1, 0)\}$ and $\{(0, 1)\}$, so the generators become unitaries, and the boundary quotient is $C^*(\mathbb{Z}^2)$.
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Q: Why only basic sets?

A: Consider the free monoid on two generators $\{a, b\}$ and let $F = \{a\}$ (which is not basic because $a \vee b = \infty$).

The corresponding relation would be $1 - V_a V_a^* = 0$, making V_a unitary in the quotient, but this would leave no room for the orthogonal range of V_b , which constitutes a 'catastrophic' failure.

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Q: Define Cuntz's $\mathcal{Q}_{\mathbb{N}}$

A: *By way of presentation:* $\mathcal{Q}_{\mathbb{N}}$ is the universal C^* -algebra generated by isometries S and $\{V_n : n \in \mathbb{N}^{\times}\}$ subject to the relations

$$(Q1) \quad V_n S = S^n V_n$$

$$(Q2) \quad V_m V_n = V_n V_m,$$

$$(Q3) \quad V_m^* V_n = V_n V_m^*, \text{ when } (m, n) = 1$$

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That was too fast; can we see a concrete example?

A concrete representation: the basic operations “add 1” and “multiply by n ” determine operators A and M_n on the Hilbert space $\ell^2(\mathbb{Z})$:

$$A\delta_m := \delta_{m+1} \quad \text{for each } m \in \mathbb{Z}$$

and

$$M_n\delta_m = \delta_{nm} \quad \text{for each } n \in \mathbb{N}^\times \quad \text{and } m \in \mathbb{Z}.$$

Facts:

- 1 These operators satisfy Cuntz's relations (Q1–5).
- 2 A and M generate a faithful copy of $\mathcal{Q}_{\mathbb{N}}$.
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Hint 1: Somewhat surprisingly, $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^{\times})$ is a qlo.

$$(r, x) \leq (s, y) \iff x^{y-1}(s-r) \in \mathbb{N} \text{ and } x|y.$$

The (mild) surprise is that the additive order appears at **different scales**.

Hint 2: $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^{\times})$ has the required amenability and topological freeness properties.

Essentially on comparing the resulting presentation of the boundary quotient to that of $\mathcal{Q}_{\mathbb{N}}$ one gets

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Remarks:

- 1 The relation (BQ4) corresponds to the basic subset $\{(1, 1)\} \subset \mathbb{N} \times \mathbb{N}^\times$.
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The map $\mathbb{N} \rtimes \mathbb{N}^\times \rightarrow \mathbb{N}^\times \hookrightarrow \mathbb{R}_+^*$ determines σ on $\mathcal{T}_{\mathbb{N} \rtimes \mathbb{N}^\times}$:

$$\sigma_t(S) = S \quad \text{and} \quad \sigma_t(V_p) = p^{it} V_p.$$

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We are interested in the KMS states.

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$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$$

for all σ -analytic elements $a, b \in A$.

(Yes, there are enough of those σ -analytic elements.)

Remark: We will distinguish between **KMS $_\infty$ states**, which are the limits of KMS $_\beta$ states as $\beta \rightarrow \infty$ and **ground states** which are those for which the function $z \mapsto \varphi(b\sigma_z(a))$ is bounded on the upper half plane.

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Theorem (L-Raeburn)

- 1 For $\beta \in [-\infty, 1)$ there are no KMS_β states
- 2 For $\beta \in [1, 2]$ there is a unique KMS_β state φ_β given by

$$\varphi_\beta(S^m V_a V_b^* S^{*n}) = \begin{cases} a^{-\beta} & \text{if } a = b \text{ and } m = n \\ 0 & \text{otherwise .} \end{cases}$$

- 3 For $\beta \in (2, \infty]$ the extremal KMS_β states are indexed by $z \in \mathbb{T}$ and are given by $\varphi_{\beta,z}(S^m V_a V_b^* S^{*n}) =$

$$\begin{cases} \frac{1}{a\zeta(\beta-1)} \sum_{a|x|(m-n)} x^{1-\beta} z^{(m-n)/x} & \text{if } a = b \text{ and } a|(m-n) \\ 0 & \text{otherwise .} \end{cases}$$

- 4 The ground states are in affine bijection with the states ω of the Toeplitz algebra $\mathcal{T}_{\mathbb{N}}$ generated by S , and are given by

$$\varphi_{\beta,z}(S^m V_a V_b^* S^{*n}) = \begin{cases} \omega(S^m S^{*n}) & \text{if } a = b = 1. \\ 0 & \text{otherwise .} \end{cases}$$

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Key ingredients in the proof

Apart from the general machinery for Toeplitz algebras of qlo groups, one needs:

- 1 an explicit description (in terms of supernatural numbers and adeles) of the spectrum of the (commutative) C^* -algebra $B_{\mathbb{N} \times \mathbb{N} \times}$ generated by the projections $1_{(m,a)}$.
- 2 a version of the Euclidean algorithm that works in parallel to the covariance condition to “Wick-order”

$$V_b^* S^{*n} S^m V_a \quad \text{into} \quad S^\gamma V_{a'} V_{b'}^* S^{*\delta},$$

where $a = a'(a, b)$, $b = b'(a, b)$, and γ and δ are the smallest non-negative solutions of $\gamma b - \delta a = (m - n)$ when they exist –the product vanishes otherwise.

Spontaneous symmetry-breaking?

For $1 \leq \beta \leq 2$ the KMS_β states vanish at $S^m V_a V_b^* S^{*n}$ if $a \neq b$ or $m \neq n$ so they are symmetric with respect to the full dual coaction of $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ on the Toeplitz algebra.

However, for $\beta > 2$ the KMS_β states vanish if $a \neq b$ or $a \nmid (m - n)$, so they are symmetric with respect to the quotient coaction of \mathbb{Q}_+^* , viewed, more classically, as the action of the infinite torus $\widehat{\mathbb{Q}_+^*}$.

As β goes from 2^- to 2^+ , the circular symmetry $\phi_z \mapsto \phi_{\lambda z}$ at the level of KMS states is broken, but such a symmetry is not apparent at the level of the C^* -algebra.

Indeed, for $\lambda \in \mathbb{T}$, the formula in part 3 of the theorem gives $\phi_{\lambda z}(S) = \frac{\lambda z}{z(\beta-1)} = \lambda \phi_z(S) = \phi_z(\lambda S)$ yet there is no automorphisms θ_λ such that $\theta_\lambda(S) = \lambda S$, because this clashes with the relation $V_a S = S^a V_a$.

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