

Can a function and its Fourier transform both have compact support?
And other questions with the same answer.

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It is a basic fact of classical Fourier analysis that if we have a non-zero function $f \in C_c(\mathbb{R}^n)$, then its Fourier transform \hat{f} is analytic and therefore does not have compact support.

For which locally compact abelian groups G can we have a non-zero $f \in C_c(G)$ with $\hat{f} \in C_c(\hat{G})$?

Show Hewitt+Ross!

Depends on the structure theory of locally compact abelian groups.

Theorem

If G is locally compact abelian group, the following are equivalent:

- 1 G has a compact open subgroup
- 2 There is a non-zero $f \in C_c(G)$ with $\hat{f} \in C_c(\hat{G})$.

Proof: 1) \implies 2):

If G has a compact open subgroup H , then $f = \chi_H \in C_c(G)$ and $\widehat{f} = \chi_{H^\perp} \in C_c(\widehat{G})$; so (i) implies (ii).

1) \iff 2):

Suppose there is a non-zero $f \in C_c(G)$ with $\widehat{f} \in C_c(\widehat{G})$ and that U is a compact neighborhood of e .

Then there is $g \in C_c(G)$ with $gL_yf = L_yf$ for $y \in U$ and

$\phi \in C_c(\widehat{G})$ with $\phi\widehat{L_yf} = \widehat{L_yf}$ for all $y \in U$.

Hence $L_{\widehat{\phi}}M(g)$ is a compact operator and $L_{\widehat{\phi}}M(g)L_yf = L_yf$ for all $y \in U$.

This implies that

$$X(U) = \text{span}\{L_y f \mid y \in U\}$$

is finite dimensional.

Clearly $V \subset U$ implies that $X(V) \subset X(U)$.

Now take a neighborhood U s.t. $X(U)$ has minimal, positive dimension.

Take a neighborhood V of e s.t. $V = V^{-1}$ and $V^2 \subset U$.

Then $X(V) = X(U)$ and $v \in V \implies L_v X(V) \subset X(U) = X(V)$.

So $X(V)$ is invariant by the open subgroup H generated by V and $X(V) = X(H)$.

$X(V) \subset C_c(G)$, pick an orthonormal basis $\{f_i\}$ for $X(V)$.

So

$$1 = \langle f_1, f_1 \rangle = \langle \lambda(h)f_1, \lambda(h)f_1 \rangle = \sum_k \langle \lambda(h)f_1, f_k \rangle \overline{\langle \lambda(h)f_1, f_k \rangle}$$

is constant on H and in $C_0(G)$, so H must be compact.

We used the following

Theorem

If $a \in C_r^(G)$ and $f \in C_0(G)$, then $aM(f)$ is a compact operator on $L^2(G)$.*

This is well known, (Takai, Vallin - although the result is probably older).

Theorem

Suppose $a \in \mathcal{L}(G)$ and $f \in L^\infty(G)$ s.t. $M(f)a \neq 0$ has finite rank. Then G has a compact open subgroup.

One version of the Heisenberg uncertainty relations is that for f, g appropriate functions on \mathbb{R}^n one has

$$L_g M(f) = M(f) L_g$$

only if one of the functions is 0.

Theorem

Suppose $a \in C_r^(G)$ and $f \in C_0(G)$ are both non-zero s.t. $aM(f) = M(f)a$. Then G has a compact open subgroup.*

Proof.

Uses Fuglede's Theorem and the Takesaki-Tatsuuma paper on "Duality and subgroups". □

Given that G has a compact open subgroup, what does functions $f \in C_c(G)$ with $\widehat{f} \in C_c(\widehat{G})$ look like?

Definition

The polynomial functions on G is the space

$$\mathcal{P}(G) = \text{span}\{L_x(\gamma\chi_H) \mid x \in G, \gamma \in \widehat{H}\}$$

where H is a compact open subgroup of G .

Theorem

Suppose G has a compact open subgroup. Then

$$\mathcal{P}(G) = \{f \in C_c(G) \mid \widehat{f} \in C_c(\widehat{G})\}$$

*and this is the only multiplier Hopf * -subalgebra of $C_0(G)$ separating points of G .*

Everything generalizes to non abelian groups.

And with $C_r^*(G)$ replacing $C_0(G)$.

Why do we do this?

Want to describe totally disconnected quantum groups.

Note that if G is totally disconnected, then $\mathcal{P}(G) = C_c^\infty(G) =$ the space of regular functions as defined by Bruhat.

(24.30) Theorem. *A locally compact Abelian group G is topologically isomorphic with $R^n \times G_0$, where G_0 is a locally compact Abelian group containing a compact open subgroup. The group $A(G, \tau(R^n))$ of (24.29) may be taken as G_0 . If G is also topologically isomorphic with $R^m \times G_1$ and G_1 contains a compact open subgroup, then $m = n$.*

Proof. Let G_0 be the group $A(G, \tau(R^n))$ of (24.29). Let H be a compactly generated open subgroup of G_0 . By (9.8), H is topologically isomorphic with $R^a \times Z^b \times F$, where F is a compact group. If $a \geq 1$, then $R^n \times G_0$ would contain a topological isomorph of R^{n+a} , contrary to (24.29.iv). Thus we have $a = 0$. Since Z^b is discrete, F is open in $Z^b \times F$, and its topological isomorph is open in H and hence in G_0 .

We now prove the second statement. Let C_0 and C_1 be the components of the identities of G_0 and G_1 , respectively. Then plainly C_0 and C_1 are compact and the component of e in G is topologically isomorphic with both $R^n \times C_0$ and $R^m \times C_1$. Corollary (9.13) now implies that $n = m$. \square

Another application of (24.8) follows. We will use it in classifying compact monothetic groups (25.11) — (25.17).

(24.31) Theorem. *Let G be a compact Abelian group with character group X . Let m be a nonzero cardinal number. The following conditions are equivalent:*