1. (12 points)

For each of the following statements, either prove it is true, or provide a counterexample to show that it is false.

(a) If \( x, y \in \mathbb{R} \), with \( x < y \), then \( x^2 < y^2 \).

False - Take \( x = -2 < -1 = y \).

\[
(-2)^2 - x^2 = 4 \quad \land \quad y^2 = (-1)^2 = 1 \quad \Rightarrow \quad x^2 > y^2
\]

(b) If \( x, y \in \mathbb{R} \), with \( |x| < |y| \), then \( x^2 < y^2 \).

True - Since \( x \) is always less than \( y \),

\[
0 \leq |x| \quad \text{so} \quad 0 \leq |x| < |y| \quad \text{and} \quad |x| < |y| \quad \Rightarrow \quad x^2 < y^2.
\]

(c) If \( S \) is a nonempty bounded subset of \( \mathbb{R} \) containing both its maximum and its minimum element, then \( S \) is a compact subset of \( \mathbb{R} \).

False - Take \( S = [1, 2) \cup (3, 4] \)

\[
\min S = 1 \in S \quad \land \quad \max S = 4 \in S
\]

But by the Heine-Borel Theorem, \( S \) is not compact since \( S \) is not closed.

\( S \) must be both closed and bounded to be compact.
2. (18 points)

(a) Let \( x \in \mathbb{R} \) and fix \( \varepsilon > 0 \). Define the notions of the \( \varepsilon \)-neighborhood of \( x \), \( N(x; \varepsilon) \), and the deleted \( \varepsilon \)-neighborhood of \( x \), \( N^*(x; \varepsilon) \).

\[
\varepsilon \text{-neighborhood of } x, \quad N(x; \varepsilon) = \{ y \in \mathbb{R} : |y - x| < \varepsilon \}
\]

\[
\text{deleting } \varepsilon \text{-neighborhood of } x, \quad N^*(x; \varepsilon) = \{ y \in \mathbb{R} : |y - x| < \varepsilon \} \setminus \{ x \}
\]

(b) Let \( S \) be a subset of \( \mathbb{R} \). Define what it means for \( x \in \mathbb{R} \) to be an accumulation point of \( S \), i.e. what does it mean to write \( x \in S' \)?

We say \( x \) is an accumulation point of \( S \) (written \( x \in S' \)) if for every \( \varepsilon > 0 \)

\[
N^*(x; \varepsilon) \cap S \neq \emptyset,
\]

i.e. if for every \( \varepsilon > 0 \) there exists \( y \in S \) with \( 0 < |y - x| < \varepsilon \).

(c) Compute the set of accumulation points of the set \( S = \{ \frac{1}{n} : n \in \mathbb{N} \} \). Justify your answer.

We claim \( S' = \\{ 0 \} \).

First we show \( 0 \in S' \).

Let \( \varepsilon > 0 \) be given. Recall from part (a),

\[
N^*(0; \varepsilon) = (-\varepsilon, 0) \cup (0, \varepsilon)
\]

By Archimedean Principle, \( \exists n \in \mathbb{N} \) with \( \frac{1}{n} < \varepsilon \). Since \( 0 < \frac{1}{n} \),

we have \( \frac{1}{n} \in N^*(0; \varepsilon) \cap S \). So \( N^*(0; \varepsilon) \cap S \neq \emptyset \) for every \( \varepsilon > 0 \).

So \( 0 \in S' \).

If \( x \neq 0 \), take \( \varepsilon = \frac{|x|}{2} > 0 \). In fact \( N^*(x; \varepsilon) \cap S = \emptyset \) and \( x \notin S' \).

If \( x > 0 \) and \( x \notin S \), let \( s = \min \{ x - y : y \in S \} \). Then \( x - s \notin \mathbb{N} \) and \( x - s \notin S \).

If \( x = \frac{1}{n} \in S \), take \( \varepsilon = \frac{1}{n+1} \). Then \( N^*(x; \varepsilon) \cap S = \emptyset \) and \( x \notin S' \).

\( \therefore S' = \{ 0 \} \).
3. (12 points)

Let \( S \) be a non-empty bounded subset of the real numbers, with \( m = \inf(S) \). Define

\[
3S = \{3 \cdot s : s \in S\}.
\]

(a) Prove that \( 3 \cdot m \) is a lower bound for \( 3S \).

\[
\text{Since } m = \inf(S), \text{ } m \text{ is a lower bound for } S.
\]

\[
i.e. \quad m \leq a, \quad \forall a \in S.
\]

\[
3 > 0 \quad \Rightarrow \quad 3m \leq 3a, \quad \forall a \in S,
\]

\[
\Rightarrow 3m \leq y, \quad \forall y \in 3S,
\]

\[
3m \text{ is a lower bound for } 3S.
\]

(b) Prove that

\[
3 \cdot m = \inf(3S).
\]

Take \( \ell \) \( \neq 3m \), we will show \( \ell \) is not a lower bound for \( 3S \).

Consequently, \( 3m \) will be the greatest lower bound for \( 3S \), i.e., \( 3m = \inf(3S) \).

\[
3m < \ell, \quad \text{Note } 0 < S, \quad \text{so } 0 < \frac{1}{3}.
\]

\[
\therefore \frac{1}{3} \cdot 3m < \frac{1}{3} \cdot \ell, \quad \text{i.e., } m < \frac{1}{3} \ell.
\]

\[
m = \inf(S) \text{ (greatest lower bound for } S \text{).} \quad \text{So } \frac{1}{3} \ell
\]

is not a lower bound for \( S \), i.e., \( S \) cannot contain

\[
0 < \frac{1}{3} \ell \quad \Rightarrow \quad 3 \cdot 0 < 3 \cdot \frac{1}{3} \ell = \ell \quad \therefore \quad 3 \cdot \ell \text{ is not a lower bound for } S.
\]

\[
3m = \inf(3S),
\]
4. (13 points)

Prove using the definition of convergence of sequences that

\[ \lim_{n \to \infty} \frac{8n^3 + 5}{4n^3 - n} = 2. \]

We need to show: given \( \varepsilon > 0 \), there exists \( N > 0 \) such that if \( n > N \),

\[ \left| \frac{8n^3 + 5}{4n^3 - n} - 2 \right| < \varepsilon \]

\[ \left| \frac{8n^3 + 5}{4n^3 - n} - 2 \right| = \left| \frac{8n^3 + 5 - 2(4n^3 - n)}{4n^3 - n} \right| \]

\[ = \left| \frac{2n + 5}{4n^3 - n} \right| = \frac{2n + 5}{4n^3 - n} \]

We note for all \( n \geq 1 \),

\[ 2n + 5 \leq 2n + 5n = 7n \]

For all \( n \in \mathbb{N} \), \( n \leq n^3 \Rightarrow -n \geq -n^3 \)

\[ 4n^3 - n \geq 4n^3 - n^3 = 3n^3 \geq 0 \]

For all \( n \in \mathbb{N} \),

\[ \frac{1}{4n^3 - n^3} \leq \frac{1}{3n^3} \]

\[ \left| \frac{2n + 5}{4n^3 - n} \right| = \left| \frac{2n + 5}{4n^3 - n} \right| = \frac{2n + 5}{4n^3 - n} \leq \frac{7}{3n^3} = \frac{7}{3n^2} \]

Take \( N = \frac{\sqrt{3\varepsilon}}{\varepsilon} \) i.e. \( \frac{17}{3\varepsilon} < N^2 \) i.e. \( N > \sqrt{\frac{17}{3\varepsilon}} \)

Then if \( n > N \),

\[ \left| \frac{8n^3 + 5}{4n^3 - n} - 2 \right| = \left| \frac{2n + 5}{4n^3 - n} \right| \leq \frac{7}{3n^2} \leq \frac{7}{3N^2} < \varepsilon, \]