EAST COAST OPERATOR ALGEBRAS SYMPOSIUM

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CLOSE SEPARABLE NUCLEAR C^*-ALGEBRAS

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THE KADISON KASTLER METRIC

Definition 1. Let $A, B \subseteq B(H)$ be C^*-algebras. We say that $d(A, B)$ is the Hausdorff distance between their unit balls. $d(A, B) < \gamma$ if and only if to each $x \in \text{Ball}(A)$, there exists $y \in \text{Ball}(B)$ with $\|x - y\| < \gamma$, and vice versa.
□

Simplification: A, B are unital with unit I_H.

Example 2. (i) u a unitary in $B(H), A \subseteq B(H)$ a C^*-algebra, $B = uAu^*$. If $\|u - I\| \leq \alpha \ll 1$, so u is close to I, then

$$d(A, B) \leq 2\alpha \ll 1.$$

(ii) $t \in B(H)$ is close to I, so invertible. $B = tAt^{-1}$.

IF B is a C^*-algebra, then A is close to B.

QUESTIONS

If A and B are close C^*-algebras, several questions arise.

Q1. Do A and B share many properties?

YES, in particular separability and nuclearity.

Q2. Are A and B isomorphic?

YES for injective von Neumann algebras,

YES for separable AF-algebras, continuous trace.

Q3. Is there a unitary $u \in B(H)$ so that $uAu^* = B$?

YES when the answer to Q2 is known to be YES.

Q4. If there is a unitary in Q3, can it be taken close to I_H?

YES for injectives, separable unital continuous trace,

NO in general.

FLAVOUR

$A, B \subseteq B(H)$, close. Take a projection $p \in A$. Choose $x \in \text{Ball}(B)$ close to p. Self-adjoint or replace by $(x + x^*)/2$. Use $\sim$ to denote closeness.

$$x^2 \sim p^2 = p \sim x$$

so $\|x - x^2\|$ is small.

$$\sigma(x) \subseteq [0, \varepsilon] \cup [1 - \varepsilon, 1]$$

for some small ε. Let q be the spectral projection for $[1 - \varepsilon, 1]$, close to x so close to p.

Consider a unitary $u \in A$, choose $x \in \text{Ball}(B)$ with $x \sim u$. Then $x^*x \sim u^*u = 1$ so x^*x is invertible and $(x^*x)^{-1/2} \sim 1$. Also $u^*x \sim 1$ so x is invertible. Put $v = x(x^*x)^{-1/2}$. Then v is a unitary in B and

$$v \sim x \sim u.$$

Let $A, B \subseteq B(H)$ be close, $d(A, B) < 1/2$. Then an easy calculation with the weak operator topology gives

$$d(A'', B'') \leq d(A, B).$$

If A is a von Neumann algebra then $A = A''$ so

$$\begin{aligned} d(B, B'') &\leq d(B, A) + d(A, B'') \\ &= d(B, A) + d(A'', B'') \\ &\leq 2d(A, B) < 1. \end{aligned}$$

Since $B \subseteq B''$, general Banach space facts give $B = B''$ so B is a von Neumann algebra.

BACKGROUND

Definition 3. A von Neumann algebra $M \subseteq B(H)$ is injective if there is a norm one projection (conditional expectation) $\mathbb{E}\colon\ B(H) \to M$. $\square$

This has many equivalent formulations.

(i) M is the weak closure of a union of finite dimensional C^*-algebras (hyperfinite).

(ii) Given operator systems $E \subseteq F$, every completely positive map $\phi\colon\ E \to M$ extends to a completely positive map $\psi\colon\ F \to M$.

(iii) For every dual normal M-bimodule X, every derivation $\delta\colon\ M \to X$ is inner ($\delta(m) = mx_0 - x_0 m$ for some $x_0 \in X$).

NUCLEAR ALGEBRAS

Definition 4. A is nuclear if, for every C^*-algebra B, $A \otimes B$ has a unique C^*-norm. □

This has many equivalent formulations.

(i) A^{**} is injective.

(ii) There exist approximate point norm factorizations of the identity map by completely positive maps through matrix algebras

$$\begin{array}{ccc} & \mathbb{M}_{n_\lambda} & \\ \phi_\lambda \nearrow & & \searrow \psi_\lambda \\ A & \xrightarrow[I]{\qquad} & A \end{array}$$

$$\lim_{\lambda} \|\psi_\lambda(\phi_\lambda(x)) - x\| = 0, \quad x \in A.$$

(iii) A is amenable.

For all dual Banach A-bimodules X, every derivation $\delta\colon\ A \to X$ is inner ($H^1(A, X) = 0$).

ALGEBRAS CLOSE TO NUCLEARS

Another characterization of nuclearity is:

(iv) For all representations $\pi\colon A \to B(H_\pi)$, the weak closure $\pi(A)''$ is injective.

If A and B are close C*-algebras of C, with A nuclear, take a representation

$$\pi\colon B \to B(H).$$

There is a representation $\rho\colon C \to B(K)$ with $H \subseteq K$ and a projection $p \in \rho(B)'$ so that $\pi(B) = \rho(B)p$. Then $\rho(A)''$ and $\rho(B)''$ are close, and $\rho(A)''$ is injective, so $\rho(B)''$ is injective. Hence $\pi(B)'' = \rho(B)''p$ is injective, so B is nuclear.

CONCLUSION: Algebras close to nuclears are nuclear.

VIRTUAL DIAGONALS

Definition 5. A virtual diagonal for A is an element $\omega \in (A\widehat{\otimes}A)^{**}$ satisfying

(i) $\omega a = a\omega,\ a \in A$;

(ii) $m^{**}(\omega)a = a,\ a \in A$, where $m\colon\ A\widehat{\otimes}A \to A$ is the multiplication map $m(a \otimes b) = ab$ ($m^{**}(\omega) = 1$ for C^*-algebras).

Theorem 6. *The following are equivalent for a C^*-algebra A*

(i) *A is nuclear,*

(ii) *A is amenable,*

(iii) *A has a virtual diagonal.*

APPROXIMATE DIAGONALS

Definition 7. An approximate diagonal is a bounded net (x_α) in $A\widehat{\otimes}A$ such that

(i) $\lim_\alpha \|ax_\alpha - x_\alpha a\| = 0,\ a \in A,$

(ii) $\lim_\alpha \|m(x_\alpha)a - a\| = 0,\ a \in A$, where $m\colon\ A\widehat{\otimes}A \to A$ is the multiplication map $m(a \otimes b) = ab$.

Theorem 8. *For C^*-algebras, the following are equivalent*

(i) *A is nuclear,*

(ii) *A is amenable,*

(iii) *A has an approximate diagonal, which can be chosen from*

$$\operatorname{conv}\{a^* \otimes a\colon\ a \in A,\ \|a\| \leq 1\}.$$

NEGATIVE RESULTS

Example 9. There exist pairs (A_n, B_n) of nonseparable nuclear C^*-algebras so that $\lim_{n\to\infty} d(A_n, B_n) = 0$ but A_n is not isomorphic to B_n.

Example 10. There exist isomorphic pairs (A_n, B_n) of non-unital separable nuclear C^*-algebras so that

$$\lim_{n\to\infty} d(A_n, B_n) = 0$$

but any isomorphism $\phi_n\colon\ A_n \to B_n$ satisfies $\|\phi_n - I\| \geq 1/70$. There are unitaries so that $u_n A u_n^* = B$, but $\|u_n - I\| \geq 1/140$ for all possible choices.

MAIN THEOREM

Theorem 11. *Let A and B be C^*-subalgebras of $B(H)$ for a separable Hilbert space H. Suppose that A is separable and nuclear, and that*

$$d(A,B) < \gamma < 10^{-12}.$$

Then

(i) *A and B are isomorphic.*

(ii) *Given finite subsets $X \subseteq \mathrm{Ball}(A), Y \subseteq \mathrm{Ball}(B)$ an isomorphism $\alpha\colon\ A \to B$ can be chosen so that*

$$\|\alpha(x) - x\| \le 40\ \gamma^{1/2}, \qquad x \in X,$$
$$\|\alpha^{-1}(y) - y\| \le 1942\ \gamma^{1/2}, \quad y \in Y.$$

(iii) *There exists a unitary $u \in (A \cup B)''$ so that $uAu^* = B$.*

CONSEQUENCE

Once we have $uAu^* = B$, write $u = e^{ih}$ and

$$A_t = e^{ith} A e^{-ith}, \qquad 0 \leq t \leq 1.$$

Then $A_0 = A$, $A_1 = B$ and we have a continuous path of isomorphic algebras perturbing A into B.

IDEA

$$\begin{array}{ccccccccc} & & B & \longrightarrow & B & & & & \\ & \nearrow & & \searrow \quad \nearrow & & \searrow & & & \cdots \\ A & & \longrightarrow & A & \longrightarrow & & A & & \end{array}$$

(i) Horizonal maps are the identity.

(ii) All maps are completely positive contractions.

(iii) The triangles approximately commute on increasing finite subsets whose unions are dense in A and B.

(iv) The maps behave as approximate $*$-homomorphisms on increasing finite subsets.

(v) The northeast maps converge point norm to $\alpha\colon A \to B$, the southeast to $\alpha^{-1}\colon B \to A$.

HOW TO START

Fix a finite self-adjoint set $X \subseteq \mathrm{Ball}(A)$. Pick γ' to satisfy $d(A, B) < \gamma' < \gamma$. Fix $\varepsilon > 0$ with $2\gamma' + \varepsilon < 2\gamma$. Let

$$\widetilde{X} = X \cup \{xx^*\colon\ x \in X\},$$

a finite set.

For each $x \in \widetilde{X}$ pick $y \in \mathrm{Ball}(B)$ with $\|x - y\| < \gamma'$ and let $Y \subseteq \mathrm{Ball}(B)$ be the union of these y's. Choose a diagram

$$\begin{array}{ccc} & \mathbb{M}_n & \\ \phi \nearrow & & \searrow \psi \\ B & \longrightarrow & B \end{array}$$

with

$$\|\psi(\phi(y)) - y\| < \varepsilon, \qquad y \in Y.$$

Extend ϕ to $\tilde{\phi}\colon\ B(H) \to \mathbb{M}_n$ and restrict to A. Put $\theta\colon\ A \to B$ to be $\psi \circ \tilde{\phi}|_A$.

$$\|\theta(\tilde{x}) - \tilde{x}\| \leq 2\gamma' + \varepsilon < 2\gamma, \qquad \tilde{x} \in \widetilde{X}.$$

We get

$$\|\theta(x_1 x_2) - \theta(x_1)\theta(x_2)\| < 6\gamma, \qquad x_i \in X.$$

Definition 12. Given C^*-algebras A, D, a finite set $X \subseteq \text{Ball}(A)$ and $\delta > 0$, we say that a completely positive contraction $\phi\colon A \to D$ is an (X, δ)-approximate $*$-homomorphism if

$$\|\phi(xx^*) - \phi(x)\phi(x^*)\| \leq \delta, \qquad x \in X.$$

Given two completely positive maps $\phi, \psi\colon A \to D$, we write

$$\phi \approx_{X,\delta} \psi$$

to mean

$$\|\phi(x) - \psi(x)\| \leq \delta, \qquad x \in X.$$

TWO APPROXIMATION LEMMAS

Lemma 13. *Let A and D be C^*-algebras with A nuclear. Given a finite set $X \subseteq \mathrm{Ball}(A)$ and $\varepsilon > 0$, and $\mu \in (0, 1/100)$, there exists a finite set $Y \subseteq \mathrm{Ball}(A)$ such that:*

if

$$\phi\colon\ A \to D$$

is a (Y, γ)-approximate $*$-homomorphism for some $\gamma < 1/17$ then there exists an (X, ε)-approximate $*$-homomorphism $\psi\colon\ A \to D$ with

$$\|\phi - \psi\| \leq 16\gamma^{1/2} + 8\mu.$$

Using the approximate diagonal, choose $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$ and $a_i \in \mathrm{Ball}(A)$ so that $\sum_{i=1}^{n} \lambda_i a_i^* a_i$ is almost 1 and $\sum_{i=1}^{n} \lambda_i (a_i^* \otimes a_i)$ almost commutes with $x \in X$. Then

$$Y = \{a_i, a_i^*\colon\ 1 \leq i \leq n\}.$$

Lemma 14. *Let A be nuclear and let D be a C^*-algebra. Given $\varepsilon > 0$ and a finite set $X \subseteq \mathrm{Ball}(A)$, there exist a finite set $Y \subseteq \mathrm{Ball}(A)$ and $\delta > 0$ with the following property:*

If

$$\phi_1, \phi_2 \colon \ A \to D$$

are (Y, δ)-approximate $*$-homomorphisms such that

$$\phi_1 \approx_{Y,\gamma} \phi_2$$

for some $\gamma \leq 2/25$, then there is a unitary $u \in D$ satisfying $\|u - 1\| < 4\gamma + 10\delta$ and

$$\phi_1 \approx_{X,\varepsilon} \mathrm{Ad}(u) \circ \phi_2 = u\phi_2 u^*.$$

Again, from the approximate diagonal for A we get $\sum_{i=1}^{n} \lambda_i = 1$ and $a_i \in \mathrm{Ball}(A)$ with

$$\left\| \sum_{i=1}^{n} \lambda_i a_i^* a_i - 1 \right\| < \delta$$

and

$$\left\| \sum_{i=1}^{n} \lambda_i x(a_i^* \otimes a_i) - \sum_{i=1}^{n} \lambda_i (a_i^* \otimes a_i) x \right\|_{A \widehat{\otimes} A} < \delta^{1/2},\ x \in X.$$

for a suitably small choice of δ.

$$Y = \{a_1, \ldots, a_n\}.$$

A KAPLANSKY LEMMA

Lemma 15. *Let* $A \subseteq B(H)$ *be a nuclear* C^**-algebra with strong closure* $M = A''$*. Let* $X \subseteq \mathrm{Ball}(A)$ *be a finite subset and let* $0 < \alpha < 2$*. Given* $\varepsilon, \mu > 0$ *there exists a finite subset* $Y \subseteq \mathrm{Ball}(A)$ *and* $\delta > 0$ *with the following property:*

Given a finite set $S \subseteq \mathrm{Ball}(H)$ *and a unitary* $u \in M$ *satisfying* $\|u - 1\| \leq \alpha$ *and*

$$\|uy - yu\| < \delta, \qquad y \in Y,$$

there exists a unitary $v \in A$ *such that* $\|v - 1\| \leq \alpha$*,*

$$\|vx - xv\| < \varepsilon, \qquad x \in X,$$

and

$$\|v\xi - u\xi\|, \|v^*\xi - u^*\xi\| < \mu, \quad \xi \in S.$$

Y, δ depend on $A, \varepsilon, \mu, \alpha$ but not on u or S. If $X = \emptyset$ then Y can be $\emptyset$ and then this is Kaplansky's $*$-strong density of $U(A)$ in $U(M)$.