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Math 2001 - Final Exam
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1. Suppose P, Q are statements. Using any method, prove that

$$\neg\Big(\big[P \wedge (P \Rightarrow Q)\big] \wedge \neg Q\Big)$$

is a tautology (i.e., is true no matter what the statements are).

Solution: We have $P \Rightarrow Q = \neg(P \wedge \neg Q) = (\neg P) \vee Q$.

Therefore

$$P \wedge (P \Rightarrow Q) = P \wedge \big((\neg P) \vee Q\big) = \big(P \wedge (\neg P)\big) \vee (P \wedge Q) = \text{FALSE} \vee (P \wedge Q) = P \wedge Q.$$

Next we have

$$(P \wedge Q) \wedge \neg Q) = P \wedge (Q \wedge \neg Q) = P \wedge \text{FALSE} = \text{FALSE}.$$

Finally the statement we have is the negation of FALSE, which is TRUE.

Another way to do it is to write out a truth table.

2. Let A be a set.

   (a) What properties must a relation R on A have in order to be a *function* $f : A \to A$?
   Solution: If $(a, b) \in R$ and $(a, c) \in R$, then $b = c$. Also for every $a \in A$ there should be a $b \in A$ such that $(a, b) \in R$.

   (b) What properties must a relation R on A have in order to be an *equivalence relation* on A?
   Solution: It should be reflexive (i.e., $(a, a) \in R$ for all $a \in A$). It should be symmetric (i.e., if $(a, b) \in R$ then $(b, a) \in R$). And it should be transitive (i.e., if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$).

   (c) Define a relation on the set $A = \{1, 2, 3, 4, 5\}$ that is BOTH a function and an equivalence relation.
   Solution: Since the relation must be reflexive, we must have $(a, a) \in R$ for every $a \in \{1, 2, 3, 4, 5\}$. But we can't have more than one ordered pair that has the same first entry, so this is *all* we can have. So the function is $f(x) = x$, the identity.

3. Suppose $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$.

   (a) How many distinct one-to-one functions $f: A \to B$ are there? Explain your answer.
   Solution: We have four choices for $f(1)$, then three choices for $f(2)$, and two choices for $f(3)$. So in total there are $4 \times 3 \times 2 = 24$ such functions.

   (b) How many distinct onto functions $f: A \to B$ are there? Explain your answer.
   Solution: There aren't any, by the pigeonhole principle. If $|A| < |B|$, then no function can ever be onto B.

4. Consider the following theorem and its "proof."

Theorem: Every integer is divisible by 2 or 3.

Proof: Assume, for the sake of contradiction, that every integer is not divisible by 2 or 3. Since 6 is an integer, it follows that 6 is not divisible by 2 or not divisible by 3. But 6 is divisible by both 2 and 3, which is a contradiction. $\Rightarrow\Leftarrow$

What is wrong with this proof?

Solution: The negation is wrong. The negation of "every integer satisfies P" is "there is at least one integer not satisfying P," not that "every integer fails to satisfy P."

5. Prove the following statement by the method of contradiction:
"For all primes a and b, if $a+b$ is prime, then $a=2$ or $b=2$." (You may use the facts that every integer is even or odd but not both, the sum of two odd integers is even, and 2 is the only even prime.)

Solution: Assume, to get a contradiction, that a and b are primes such that $a+b$ is prime, while $a \neq 2$ and $b \neq 2$.

Since the only even prime is 2, the fact that a is prime and not 2 means that a is not even. So a is odd. Similarly b is odd.

Now $a+b$ is the sum of two odd integers, and therefore it is even. Since $a>1$ and $b>1$, we know that $a+b>2$, so $a+b$ must be odd. The fact that $a+b$ is both even and odd is impossible, so we get a contradiction.

6. Define a sequence $a_0, a_1, a_2, \ldots$ by $a_0 = 1$ and

$$a_{n+1} = 1 - \frac{1}{4a_n}, \qquad n \geq 0.$$

Prove either by induction or by the method of smallest counterexample that

$$a_n = \frac{2+n}{2+2n}.$$

Solution: Let's do proof by induction. First the base case: when $n=0$ we have $a_0 = 1$ and $\frac{2+0}{2+0} = 1$, so it works.

Now suppose $a_n = \frac{2+n}{2+2n}$; we want to prove that $a_{n+1} = \frac{3+n}{4+2n}$. We have

$$a_{n+1} = 1 - \frac{2+2n}{8+4n} = 1 - \frac{1+n}{4+2n} = \frac{3+n}{4+2n},$$

which is what we wanted to see. So the formula is valid for all n.

7. Consider the following permutations in S_7:

$$\sigma = (1,4,3)(2,7,5,6), \qquad \tau = (1)(2,3,5,7)(4,6).$$

(a) Compute $\sigma \circ \tau$.
Solution: $\sigma \circ \tau = (1,4,2)(3,6)(5)(7)$.

(b) Write σ as a product of transpositions, and determine whether σ is even or odd.
Solution: We have $\sigma = (1,4) \circ (4,3) \circ (2,7) \circ (7,5) \circ (5,6)$. That's five transpositions, so σ is odd.

8. Let A, B be events in a sample space, and suppose that A and B are independent.

(a) Express $P(A \cup B)$ in terms of $P(A)$ and $P(B)$. (Note that this question is asking for the probability of the *union*, not the intersection!)
Solution: We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Since A and B are independent, we know $P(A \cap B) = P(A)P(B)$. So

$$P(A \cup B) = P(A) + P(B) - P(A)P(B).$$

(b) Express $P(\overline{A} \cap \overline{B})$ in terms of $P(A)$ and $P(B)$.
Solution: This event is the complement of $A \cup B$, by DeMorgan's Law, so that

$$P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A)P(B).$$

(c) Prove that $\overline{A}$ and $\overline{B}$ are independent.
Solution: We have $P(\overline{A}) = 1 - P(A)$ and $P(\overline{B}) = 1 - P(B)$, so that

$$P(\overline{A}) \cdot P(\overline{B}) = (1 - P(A))(1 - P(B)) = 1 - P(A) - P(B) + P(A)P(B),$$

which is the same thing we found for $P(\overline{A} \cap \overline{B})$.

9. A set of five cards is called a "flush house" if it has three cards of one suit and two cards of another suit. For example, the 3, 4, and king of diamonds and the 7 and ace of hearts would form a flush house.

What is the probability of getting a flush house from a full deck?

Solution: List the ways to choose the cards.

- First choose the suit for the triplet; there are $\binom{4}{1}$ ways to do this.
- Next choose the three faces in the triplet; there are $\binom{13}{3}$ ways to do this.
- Now choose the suit for the pair; there are $\binom{3}{1}$ ways to do this.
- Finally choose the two faces for the pair; there are $\binom{13}{2}$ ways to do this.

Since the size of the sample space is $\binom{52}{5}$, and since every hand is equally likely, we have

$$P = \frac{\binom{4}{1}\binom{13}{3}\binom{3}{1}\binom{13}{2}}{\binom{52}{5}} = \frac{429}{4165}.$$

10. An unfair coin obtained from Tom's Discount Word Problem Supply Shop has probability $\frac{2}{3}$ of coming up heads. Suppose the coin is tossed three times.

(a) What is the probability that it comes up tails exactly two times?
Solution: There are three ways this could happen: we get (H, T, T) or (T, H, T) or (T, T, H). The probability of each one of these is $\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{27}$. Thus the probability of any one of the three pairwise disjoint events is $\frac{2}{9}$.
$0.9 \cdot 0.1 \cdot 0.1 = 0.009$. So the total probability is 0.027.

(b) Given that at least one of the tosses is tails, what's the probability it comes up tails all three times?
Solution: The set B consisting of at least one tails is the complement of the set $\overline{B} = \{(H, H, H)\}$, which occurs with probability $(\frac{2}{3})^3 = \frac{8}{27}$. So the probability of B is $P(B) = 1 - \frac{8}{27} = \frac{19}{27}$.
The intersection $A \cap B$ is just the one element (T, T, T), and $P(A \cap B) = (\frac{1}{3})^3 = \frac{1}{27}$. So the conditional probability is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{27}}{\frac{19}{27}} = \frac{1}{19}.$$

11. (a) Prove that the integers $a = 13, b = 5$ are relatively prime by using Euclid's algorithm to compute $\gcd(13, 5)$.
Solution: Write $13 = 2 \cdot 5 + 3$, then $5 = 1 \cdot 3 + 2$, and then $3 = 1 \cdot 2 + 1$.
We have $\gcd(13, 5) = \gcd(5, 3) = \gcd(3, 2) = \gcd(2, 1) = 1$.

(b) Find integers x and y such that

$$13x + 5y = 1.$$

Solution: We go backwards:

$$\begin{aligned}
1 &= 3 - 2 \\
1 &= 3 - (5 - 3) \\
&= 2 \cdot 3 - 5 \\
1 &= 2 \cdot (13 - 2 \cdot 5) - 5 \\
&= 2 \cdot 13 - 5 \cdot 5.
\end{aligned}$$

So $x = 2$ and $y = -5$.

(c) Find 5^{-1} in $\mathbb{Z}_{13}$.

Solution: Since $1 = 2 \cdot 13 - 5 \cdot 5$, we have $1 = (\ominus 5) \otimes 5$ in $\mathbb{Z}_{13}$. So $5^{-1} = \ominus 5 = 8$ in $\mathbb{Z}_{13}$.

(d) Compute $2 \oslash 5$ in $\mathbb{Z}_{13}$.

Solution: We have $2 \oslash 5 = 2 \otimes 5^{-1} = 2 \otimes 8 = 3$.

12. (a) Find all solutions in $\mathbb{Z}_6$ of the quadratic equation $x^2 \ominus 3x \oplus 2 = 0$ by testing each element of $\mathbb{Z}_6$ directly.

Solution: We have:

0: $x^2 \ominus 3x \oplus 2 = 2$.
1: $x^2 \ominus 3x \oplus 2 = 0$.
2: $x^2 \ominus 3x \oplus 2 = 0$.
3: $x^2 \ominus 3x \oplus 2 = 2$.
4: $x^2 \ominus 3x \oplus 2 = 0$.
5: $x^2 \ominus 3x \oplus 2 = 0$.

(b) Why does writing the equation as $(x \ominus 1) \otimes (x \ominus 2) = 0$ and concluding that $x = 1$ or $x = 2$ not give you all the solutions?

Solution: Because 6 is not prime, we cannot say $a \otimes b = 0$ in $\mathbb{Z}_6$ implies that a or b have to be zero. For example $2 \otimes 3 = 0$ in $\mathbb{Z}_6$.