

**Classification of Inductive Limit
 C^* -algebras**

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(Part of the talk is based on the joint work
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H — complex Hilbert space

$B(H)$ — algebra of bounded operators on H

C^* -algebra — norm closed subalgebra $\mathcal{A} \subset B(H)$

such that $T \in \mathcal{A} \Leftrightarrow T^* \in \mathcal{A}$.

Example 1. $M_n(\mathbb{C})$ — $n \times n$ matrices on \mathbb{C} , acting on $H = \mathbb{C}^n$, i.e., $M_n(\mathbb{C}) = B(H)$.

Example 2. $C(X)$ — continuous complex valued functions on compact Hausdorff space X , acting on $L^2(X, \mu)$, $f \mapsto M_f$ with $M_f(g) = fg$, $\forall g \in L^2(X, \mu)$. $C(X) \subset B(L^2(X, \mu))$.

Example 3. $M_n(C(X))$ — $n \times n$ matrices over $C(X)$.

Example 4. If $\mathcal{A} \subset B(H)$ is a C^* -algebra, then

$$M_n(\mathcal{A}) \subset M_n(B(H)) \cong B(H \oplus H \oplus \dots \oplus H)$$

Simple C^* -algebra: A C^* -algebra without any non-trivial proper closed two sided ideals.

C^* -algebras: Noncommutative spaces.

$$X \quad \longleftrightarrow \quad C(X)$$

topological space

commutative C^* -algebra

Some geometrical or topological objects correspond to C^* -algebras which are not commutative.

Applications (cf Paul Baum's talk): Novikov Conjecture about homotopy invariance of the high signature of manifolds, Gromov-Lawson-Rosenberg Conjecture about Positive scalar curvature (Both are consequences of Baum-Connes Conjecture)

$$\mathbb{Z} \text{ action } \alpha \text{ on space } X \quad \longleftrightarrow \quad C(X) \rtimes_{\alpha} \mathbb{Z}$$

($\alpha: X \rightarrow X$ homeomorphism) (cross product C^* -algebra)

Foliation $(V, F) \longrightarrow C_r^*(V, F)$ (Foliation C^* -algebra).

Noncommutative quotient space (Connes):

$X = \{x, y\}$, \sim identifying x and y

$X/\sim = \{pt\}$, $C(X/\sim) = \mathbb{C}$.

Noncommutative quotient:

$$C(X) = \mathbb{C} \oplus \mathbb{C} \longrightarrow M_2(\mathbb{C}) \quad \text{by } (a, b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$x \sim y \text{ means } p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \sim q := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The partial isometry $v := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ has initial space

p and final space q .

Put v (and v^*) in $\mathbb{C} \oplus \mathbb{C}$: then $p \sim q$.

The quotient space is $M_2(\mathbb{C})$

$$M_2(\mathbb{C}) \stackrel{\text{stably}}{\cong} \mathbb{C}$$

Foliation (V, F)

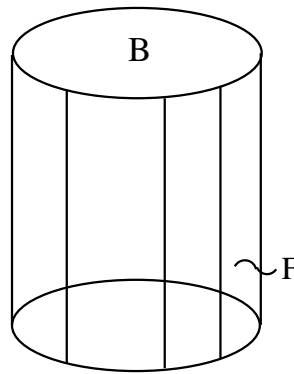
Leaf space should be the quotient space by identifying all the points in each leaf into one point.

Trivial foliation:

$$V = B \times F$$

leaf space is

$$V/F = B$$

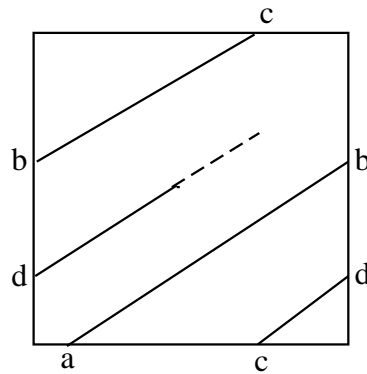


Kronecker foliation on torus T^2 with irrational slopes

every leaf is dense

V/F : non Hausdorff

$$C(V/F) = \mathbb{C}$$



Foliation algebra

is noncommutative leaf space, which can tell the difference of the Kronecker foliation (e.g., different slopes make the difference)

Most simple C^* -algebras are highly non-commutative.

Cross product: $\alpha : X \longrightarrow X$

If α is trivial

If α is minimal and free

$$(\forall x, \alpha(x) = x)$$

$$(\forall x, \overline{\{\alpha^n x; n \in \mathbb{Z}\}} = X)$$

$\forall n \neq 0, \alpha^n$ has no fixed point

$$C(X) \rtimes_{\alpha} \mathbb{Z} = C(X \times S^1)$$

$C(X) \rtimes_{\alpha} \mathbb{Z}$ is simple

Foliation

Trivial foliation $V = B \times F$

If every leaf is dense

$$C_r^*(V, F) \stackrel{\text{stably}}{\cong} C(B)$$

$C_r^*(V, F)$ is simple

Simple C^* -algebras and commutative C^* -algebras are two extremes in the theory of C^* -algebras.

But one can use $M_n(C(X))$ to construct many important simple C^* -algebras by inductive limit construction.

AH-algebra (approximately homogeneous):

$X_{n,i}$ — compact metric spaces

$P_{n,i} \in M_{[n,i]}(C(X_{n,i}))$ are projections,

$P_{n,i}^2 = P_{n,i} = P_{n,i}^*$, where $[n, i]$ are positive integers.

$\mathcal{A}_n = \bigoplus_{i=1}^{k_n} P_{n,i} M_{[n,i]}(C(X_{n,i})) P_{n,i}$

$\phi_n : \mathcal{A}_n \rightarrow \mathcal{A}_{n+1}$ homomorphism

An AH-algebra \mathcal{A} is an inductive limit

$$\mathcal{A}_1 \xrightarrow{\phi_1} \mathcal{A}_2 \xrightarrow{\phi_2} \dots \longrightarrow \mathcal{A}_n \xrightarrow{\phi_n} \dots$$

Let $\phi_{n,m} = \phi_{m-1} \circ \phi_{m-2} \circ \dots \circ \phi_n : \mathcal{A}_n \rightarrow \mathcal{A}_m$. Denote

$$\mathcal{A} = \lim(\mathcal{A}_n, \phi_{n,m}).$$

Elliott-Gong-Li: One can make each $X_{n,i}$ simplicial complex and each ϕ_n injective. In this case, $\mathcal{A}_n \xrightarrow{\phi_n} \mathcal{A}_{n+1}$ as a subalgebra. $\mathcal{A} = \overline{\bigcup_{n=1}^{\infty} \mathcal{A}_n}$, the completion.

If $\mathcal{A}_n \subset \bigoplus M_{[n,i]}(C(X_{n,i}))$, then \mathcal{A} is called approximately sub-homogeneous algebra (ASH algebra).

Many important cross product C^* -algebras and foliation C^* -algebras are simple AH-algebras.

Conjecture: All the simple, separable, nuclear, stably finite C^* -algebras are ASH-algebras.

(Bratteli, Boca, Elliott, Evans, Kishimoto, Q. Lin, H. Lin, Phillips and Putnam)

Putnam: Cross product of a minimum free \mathbb{Z} action on a 0-dimensional space is a simple AH-algebra.

Elliott & Evans: Foliation algebra of Kronecker foliations (irrational rotation algebras) are simple AH-algebras.

Q. Lin & Phillips: Cross product of a minimum \mathbb{Z} diffeomorphism on a smooth manifold is a simple ASH-algebra.

Invariant: \mathcal{A} a unital C^* -algebra.

$$V(\mathcal{A}) = \{p; p \in M_\infty(\mathcal{A}), p = p^2 = p^*\} / \sim.$$

$$p \sim q \Leftrightarrow \exists x \text{ such that } p = x^*x, q = xx^*$$

$$\text{“addition”}: [p] + [q] = [p \oplus q].$$

• $K_0(\mathcal{A})$: Grothendick group of $V(\mathcal{A})$.

$$K_0(\mathcal{A}) = \{[p] - [q]; [p] \in V(\mathcal{A}), [q] \in V(\mathcal{A})\} / \sim$$

$$[p_1] - [q_1] \sim [p_2] - [q_2] \text{ iff } \exists r \text{ such that } [p_1] + [q_2] + [r] = [p_2] + [q_1] + [r].$$

For X finite simplicial complex:

$$K_0(C(X)) = K^0(X) \xrightarrow{\text{chern map}} \cong_{\otimes \mathbb{Q}}$$

$$H^0(X) \oplus H^2(X) \oplus \dots \oplus H^{2n}(X) \oplus \dots$$

For noncommutative C^* -algebras, no such grading.

• $K_0(\mathcal{A})_+ = \{[p] - [0]\} \subset K_0(\mathcal{A})$. • $[\mathbf{1}_\mathcal{A}] \in K_0(\mathcal{A})$.

$(K_0(\mathcal{A}), K_0(\mathcal{A})_+, [\mathbf{1}_\mathcal{A}])$: scaled ordered group.

Example 1. For $\mathcal{A} = \mathbb{C}$.

$$(K_0(\mathcal{A}), K_0(\mathcal{A})_+, [\mathbf{1}_\mathcal{A}]) = (\mathbb{Z}, \mathbb{N} \cup \{0\}, 1)$$

Example 2. For $\mathcal{A} = C(S^2)$,

$$(K_0(\mathcal{A}), K_0(\mathcal{A})_+, [\mathbf{1}_\mathcal{A}]) = (\mathbb{Z} \oplus \mathbb{Z}, \mathbb{N} \oplus \mathbb{Z} \cup \{0, 0\}, (1, 0))$$

$$U(\mathcal{A}) = \{u \in \mathcal{A}; uu^* = u^*u = 1\}, \quad U_n(\mathcal{A}) = U(M_n(\mathcal{A}))$$

$$U_n(\mathcal{A}) \hookrightarrow U_{n+1}(\mathcal{A}) \quad u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

$$U_\infty(\mathcal{A}) = \cup_{n=1}^\infty U_n(\mathcal{A})$$

- $K_1(\mathcal{A}) =$ path connected components of $U_\infty(\mathcal{A})$

“addition”: $[u] + [v] = [uv]$.

- $T\mathcal{A}$ — tracial states of \mathcal{A} .

$\tau \in T\mathcal{A}$, $\tau : \mathcal{A} \rightarrow \mathbb{C}$ satisfy:

1. $\tau(ax + by) = a\tau(x) + b\tau(y)$, $\forall a, b \in \mathbb{C}$, $x, y \in \mathcal{A}$.
2. $\tau(x^*x) \geq 0 \quad \forall x \in \mathcal{A}$.
3. $\tau(xy) = \tau(yx)$.
4. $\tau(1) = 1$.

- $\langle \cdot, \cdot \rangle, T\mathcal{A} \times K_0(\mathcal{A}) \rightarrow \mathbb{C}$, $\langle \tau, x \rangle = \sum_{i=1}^n \tau(p_{ii}) - \sum_{i=1}^n \tau(q_{ii})$,

where x is represented by $[p] - [q]$ with $p = (p_{ij})_{n \times n} \in M_n(\mathcal{A})$, $q = (q_{ij})_{n \times n} \in M_n(\mathcal{A})$

Elliott invariant:

$$Inv(\mathcal{A}) = (K_0(\mathcal{A}), K_0(\mathcal{A})_+, \mathbf{1}_{\mathcal{A}}, K_1(\mathcal{A}), T\mathcal{A}, \langle \cdot, \cdot \rangle)$$

Conjecture: Elliott invariant $Inv(\mathcal{A})$ is a complete invariant for simple, nuclear, separable C^* -algebras. In particular:

Conjecture (Elliott) Let \mathcal{A} and \mathcal{B} be simple AH-algebras. Then

$$\mathcal{A} \cong \mathcal{B} \iff Inv(\mathcal{A}) \cong Inv(\mathcal{B}).$$

General assumption: $\sup \dim(X_{n,i}) < \infty$.

A. Toms has constructed counterexamples for the conjecture, if not assume $\sup \dim(X_{n,i}) < \infty$.

Real rank zero: \mathcal{A} is of real rank zero if $\forall a \in \mathcal{A}_{sa}, \forall \varepsilon > 0, \exists b \in \mathcal{A}_{sa}$ with finite spectrum such that $\|a - b\| < \varepsilon$ (denoted by $RR(\mathcal{A}) = 0$).

$$RR(C(X)) = 0 \iff \dim(X) = 0 \text{ (} X \text{ totally disconnected)}$$

Theorem (Elliott, 1993, 1995) If \mathcal{A}, \mathcal{B} are simple AH algebras with $X_{n,i} = [0, 1]$, or S^1 , then

$$\mathcal{A} \cong \mathcal{B} \Leftrightarrow \text{Inv}(\mathcal{A}) \cong \text{Inv}(\mathcal{B})$$

Theorem (Li, Memoirs of AMS, **127** (605), 1997) If \mathcal{A}, \mathcal{B} are simple AH algebras with $X_{n,i} = \text{trees}$, then

$$\mathcal{A} \cong \mathcal{B} \Leftrightarrow \text{Inv}(\mathcal{A}) \cong \text{Inv}(\mathcal{B})$$

Theorem (Li, JFA, **192** 1–51, (2002)) If \mathcal{A}, \mathcal{B} are simple AH algebras with $\dim(X_{n,i}) = 1$, then

$$\mathcal{A} \cong \mathcal{B} \Leftrightarrow \text{Inv}(\mathcal{A}) \cong \text{Inv}(\mathcal{B})$$

Theorem Elliott-Gong-Li (Invent Math., **168**, 249–320(2007)) Let \mathcal{A} and \mathcal{B} be simple AH-algebras with $\sup \dim(X_{n,i}) < \infty$. Then

$$\mathcal{A} \cong \mathcal{B} \iff \text{Inv}(\mathcal{A}) \cong \text{Inv}(\mathcal{B}).$$

Ingredients in the proof

Theorem (Li, Crelle's J. **507** 57–79 (1999)) If $A = \lim(\mathcal{A}_n, \phi_{n,m})$ is an AH algebra with $\sup_{n,i} \dim(X_{n,i}) < +\infty$, then for any finite set $F \subset \mathcal{A}_n$, $\varepsilon > 0$, there are $m > n$ and $B = \oplus_i M_{s_i}(C[0, 1])$ —direct sum of interval algebras and homomorphisms

$$\psi : \mathcal{A}_n \rightarrow \mathcal{B} \rightarrow \mathcal{A}_m$$

such that for any $\tau \in T\mathcal{A}_m$

$$\|\tau(\phi_{n,m}(f)) - \tau(\psi(f))\| < \varepsilon, \forall f \in F.$$

That is, at the level of $T\mathcal{A}$, the homomorphism can be approximated by homomorphisms factoring through \mathcal{B} — direct sum of matrix algebras over $[0, 1]$.

If $RR(\mathcal{A}) = 0$, then \mathcal{B} could be chosen to be $\oplus_i M_{s_i}(\mathbb{C})$.

That is, $[0, 1]$ can be replaced by $\{pt\}$.

Decomposition Theorem (Gong) (roughly) Suppose that \mathcal{A} is a simple AH-algebra with $\sup_{n,i} \dim(X_{n,i}) < +\infty$. For any finite set $F \subset \mathcal{A}$, one can decompose each element $a \in F$ into two parts

$$a \simeq pap \oplus a_1 \in p\mathcal{A}p \oplus (1-p)\mathcal{A}(1-p),$$

where $p \in \mathcal{A}$ is a projection and $p \ll (1-p)$, and a_1 is in a subalgebra

$$\mathcal{B} = \bigoplus_{i=1}^q M_{n_i}(C[0,1]) \subset (1-p)\mathcal{A}(1-p).$$

The case $\dim(X_{n,i}) = 1$ is due to Li (JFA). Gong uses the method for dimension one case. Li's result (Crelle's J.) also plays an essential role.

Huaxin Lin gives an abstraction of classification of EGL based on the above property. He calls it TAI property.

Theorem (Li, J. of Ramanujan Math. Soc. **21**, 365-390(2006)) If $\mathcal{A} = \lim(\mathcal{A}_n, \phi_{n,m})$ is a simple AH algebra with $\sup_{n,i} \dim(X_{n,i}) < +\infty$, then \mathcal{A} can be written as inductive limit of direct sums of matrix algebras over $\{pt\}$, $[0, 1]$, S^1 , $T_{II,k}$ and $M_k(I_n)$, where

$$I_n = \{f \in M_n(C[0, 1]); f(0) \in \mathbb{C}\mathbf{1}_n, f(1) \in \mathbb{C}\mathbf{1}_n\},$$

and $T_{II,k}$ is a connected 2-dimensional simplicial complex with $H^2(T_{II,k}) = \mathbb{Z}/k\mathbb{Z}$, $H^1(T_{II,k}) = 0$.

So the dimensions of the local spectra can be reduced to 2 or less (Gong proved that the dimensions can be reduced to 3 or less).

S. Molinas: The dimension can not be reduced to one.

To unify the classification of real rank zero AH algebras of Elliott-Gong (Ann. of Math. **144**, 497–610 (1996)) and Dadarlat-Gong (GAFA, **7**, 646-711 (1997)) and classification of simple AH algebras of Elliott-Gong-Li (Invent Math., **168**, 249-320(2007)), we consider:

C^* -algebra with ideal property: Every proper closed two sided ideal is generated by the projections inside the ideal.

All real rank zero C^* -algebras and simple C^* -algebras have ideal property.

(Pasnicu studied this class of C^* -algebras intensively.)

Stevens (1995) Classified unital AH algebras with ideal property for the case $X_{n,i} = [0, 1]$ but with an extra restriction of approximate divisibility.

Ji-Jiang (2007) Removed the restrictions of approximate divisibility and unital.

Theorem Gong-Jiang-Li-Pasnicu (2007) An AH-algebra with ideal property with uniformly bounded dimension of $X_{n,i}$ can be rewritten as AH-algebras with $Y_{n,i} = \{pt\}, [0, 1], S^1, T_{II,k}, T_{III,k}$ and S^2 , where $T_{III,k}$ is a connected 3-dimensional simplicial complex with

$$H^3(T_{III,k}) = \mathbb{Z}/k\mathbb{Z}, H^1(T_{III,k}) = 0 = H^2(T_{III,k}).$$

Ingredients in the proof: Li's result (Crelle's J.) can be generalized to this case which can be used to prove decomposition theorem for this case.