1. (15) Let $f(x) = \cosh x \;=\; \frac{e^x+e^{-x}}{2}$, $-\pi < x \leq \pi$. Consider the 2π-periodic extension of f to a function defined on all of $\mathbb{R}$.

(i) Prove that the trigonometric form of the Fourier series for f is given by

$$f(x) \;\sim\; \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx;$$

here $\sinh x = \frac{e^x-e^{-x}}{2}$.

Solution: Consider the exponential coeffiecients

$$c_n(f) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}[\frac{e^x+e^{-x}}{2}]e^{-inx}dx = \frac{1}{4\pi}\int_{-\pi}^{\pi}e^xe^{-inx}dx + \frac{1}{4\pi}\int_{-\pi}^{\pi}e^{-x}e^{-inx}dx$$

$$= \frac{1}{4\pi}[\frac{1}{1-in}e^{(1-in)x}|_{x=-\pi}^{x=\pi}+\frac{1}{1+in}e^{(-1-in)x}|_{x=-\pi}^{x=\pi}] = \frac{1}{4\pi}[\frac{(-1)^n}{1-in}(e^{\pi}-e^{-\pi})+\frac{(-1)^n}{1+in}(-e^{-\pi}+e^{\pi})]$$

$$= \frac{e^{\pi}-e^{-\pi}}{4\pi}[\frac{(-1)^n(1+in)+(1-in))}{(1-in)(1+in)} = \frac{(-1)^n(e^{\pi}-e^{-\pi})}{2\pi(1+n^2)} = \frac{\sinh\pi}{\pi}\frac{(-1)^n}{(1+n^2)}.$$

Thus for $n \geq 0$, $a_n = c_n + c_{-n} = \frac{2\sinh\pi}{\pi}\frac{(-1)^n}{(1+n^2)}$, and $a_0 = \frac{2\sinh\pi}{\pi}$. Since f is even, $b_n(f) = 0$ for all $n \geq 1$. The result follows.

(ii) By taking an appropriate value for x in part (i) above, and the fact that the fact that $\cos n\pi \;=\; (-1)^n$ for all integers n, find the sum of the infinite series

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+1)}.$$

Solution: Put $x = \pi$ and use Dirichlet's Theorem (note that $f \in PSC(\mathbb{T})$ and $f(\pi) = f(-\pi) = \cosh\pi$). Then, by Dirichlet's Theorem:

$$\frac{f(\pi-)+f(\pi+)}{2} \;=\; \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos n\pi;$$

so that

$$\cosh\pi \;=\; \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(-1)^n}{n^2+1}.$$

Hence

$$\frac{\pi}{2\sinh\pi}\cdot\cosh\pi \;=\; \frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n^2+1}.$$

Thus

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1} \;=\; \frac{\pi}{2}\coth x - \frac{1}{2}.$$

2. (15)

(i) Let f be the 2π-periodic function defined on $(-\pi, \pi)$ by $f(x) = x^2 - 2\pi x$. Prove that the trigonometric form of the Fourier series for f is given by

$$f(x) \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n[\frac{1}{n^2}\cos nx + \frac{\pi}{n}\sin nx].$$

Solution: We calculate

$$a_0(f) = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 - 2\pi x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2)dx = \frac{1}{\pi}[\frac{x^3}{3}]_{x=-\pi}^{x=\pi} = \frac{2\pi^2}{3}.$$

$$a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 - 2\pi x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx dx$$

$$= \frac{1}{\pi}[\frac{2nx\cos nx + (n^2x^2 - 2)\sin nx}{n^3}]_{x=-\pi}^{x=\pi} = \frac{4n\pi(-1)^n}{\pi n^3} = \frac{4(-1)^n}{n^2},\ n \geq 1.$$

$$b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 - 2\pi x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(-2\pi x)\sin nx dx$$

$$= -2\pi\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx dx == -2\pi\frac{2}{\pi}\int_{0}^{\pi}x\sin nx dx$$

$$= -2\pi\frac{2(-1)^{n+1}}{n} = \frac{4\pi(-1)^n}{n}.$$

The result follows.

(ii) What is the value of the Fourier series for f in part (i) above (i.e., what is the value of the sum on the right-hand side) at $x = \pi$? Be sure to justify your reasoning.

Solution: We use Dirichlet's Theorem again: Since f is piecewise smooth,

$$\frac{f(\pi-) + f(\pi)}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n[\frac{1}{n^2}\cos n\pi + \frac{\pi}{n}\sin n\pi].$$

Now $f(\pi-) = (\pi)^2 - 2\pi \cdot \pi = -\pi^2$, and $f(\pi+) = (-\pi)^2 - 2\pi \cdot (-\pi) = \pi^2 + 2\pi^2 = 3\pi^2$. So we get that the right hand side (the Fourier series) evaluated at π is equal to $\frac{-\pi^2+3\pi^2}{2} = \pi^2$.

3. (15) Consider the function f defined on $(-\pi, \pi]$ by

$$f(x) \ = \ x, \ -\pi < \ x \ < \pi, \ f(x) \ = \ 0, \ \ x \ = \ \pi.$$

Recall we have calculated that the Fourier series for f is given by

$$f(x) \ \sim \ \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n} \sin nx.$$

(i) By finding the indefinite integral of f or otherwise, determine the Fourier series for the 2π-periodic function defined by

$$g(x) \ = \ x^2, \ -\pi < \ x \ \leq \pi.$$

Solution: We note that $g(x) = \int_0^x 2f(y)dy, \ x \in [-\pi, \pi]$, and since $a_0(f) = 0$, the Fourier coefficients for g are given by $a_n(g) \ = \ \frac{-b_n(2f)}{n}, \ n \geq 1$, and $b_n(g) = \frac{a_n(2f)}{n}, \ n \geq 1$. So $a_n(g) = \ -2 \cdot 2\frac{(-1)^{n+1}}{n^2} \ = \ \frac{4(-1)^n}{n^2}, \ b_n(g) = \frac{a_n(2f)}{n} = 0, \ n \geq 1.$

We then calcuate $a_0(g)$ directly:

$$a_0(g) \ = \ \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx \ = \ \frac{2}{\pi}\int_0^{\pi} x^2 dx = \ \frac{2}{\pi}[\frac{x^3}{3}]_{x=0}^{x=\pi} \ = \ \frac{2\pi^2}{3}.$$

So our Fourier series is given by:

$$x^2 = g(x) \ \sim \ \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

(ii) Draw a graph of the values that the Fourier series for g in part (i) above takes on over the interval $(-3\pi, 3\pi]$. Be sure to explain your reasoning.

Solution: Since $g(-\pi) = g(\pi) = \pi^2$, the 2π-periodic extension of g to all $\mathbb{R}$ is in $PSC(\mathbb{T})$. It follows by a Corollary to Dirichlet's Theorem that the Fourier series just computed converges to the 2π-periodic extension of g. So on $[-\pi, \pi]$, the graph looks like a parabola, which is repeated on both the intervals $[-3\pi, -\pi]$ and $[\pi, 3\pi]$.

4. (15) Let $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$, $-\pi < x < \pi$.

(i) In homework, you calculated that the complex exponential form of the Fourier series for the 2π-periodic extension of f is given by:

$$f(x) \sim \frac{\sinh \pi \cdot i}{\pi} \sum_{n \in \mathbb{Z}} \frac{(-1)^n n}{1+n^2} e^{inx},$$

By whatever means you wish, calculate the Fourier cosine coeffiecients $a_n(f)$ for $n \geq 0$.

Solution: Note that f is an odd function: $f(-x) = f(x)$. Hence $a_n(f) = 0$ for all $n \geq 0$, and no further calculation is needed here.

(ii) Suppose we define $f(\pi) = 0$ and extend f to a 2π-periodic function defined on all of $\mathbb{R}$. Determine whether the following statement is true or false. Justify your answer.

"For $N \geq 0$, let $S_N^f(x) = \frac{(e^\pi - e^{-\pi})i}{2\pi} \sum_{n=-N}^{N} \frac{(-1)^n n}{1+n^2} e^{inx}$. Then $S_N^f(x)$ converges to $f(x)$ **uniformly** on the interval $(-2\pi, 2\pi]$."

Solution: Upon defining $f(\pi) = 0$, the 2π-periodic extension of f is piecewise smooth and averaged on $\mathbb{R}$. So, by the Corollary to Dirichlet's Theorem, $S_N^f(x)$ converges **pointwise** to $f(x)$ for all $x \in \mathbb{R}$. However, f is discontinous at $x = \pi$ and $x = -\pi$, even though it is averaged there. In fact at those points there is a jump discontinuity. It follows that Gibbs phenomenon occurs at those points, i.e. there is a sequence $\{x_N\}$ converging to π, such that $|S_N^f(x_N) - f(x_N)| \geq 0.089|f(\pi-) - f(\pi+)|$. Thus the sequence S_N^f cannot converge to f uniformly on $(-2\pi, 2\pi]$.