1. (13) For each of the following statements, either prove it is true, or provide a counterexample to show that it is false.

(i) If \((s_n)\) is a sequence such that \(\lim \inf |s_n| = 0\), then \(\lim \inf s_n = 0\).

False. Let \((s_n) = ((-1)^n - 1)\). Then \(s_n = -2\) if \(n\) is odd and \(s_n = 0\) if \(n\) is even. So \(|s_n| = 2\) if \(n\) is odd, and \(|s_n| = 0\) if \(n\) is even. Therefore \(\lim \inf |s_n| = 0\), but \(\lim \inf s_n = -2\).

(ii) If \(I\) is an interval, \(f : I \to \mathbb{R}\) is continuous on \(I\), and \((x_n)\) is a Cauchy sequence in \(I\), then \((f(x_n))\) is a Cauchy sequence in \(\mathbb{R}\).

False. Let \(I = (0, 2)\) and let \(f(x) = \frac{1}{x}\). Note \(f\) is continuous on \(I\). Now consider the Cauchy sequence \((x_n) = \left(\frac{1}{n}\right)\). Clearly each \(x_n \in I\), and \((x_n)\) is Cauchy since \((x_n)\) converges to 0 (which is not an element of \(I\)). Note that \((f(x_n)) = \left(\frac{1}{x_n}\right) = (n)\), and the sequence \((n)\) diverges to \(+\infty\), thus is not a Cauchy sequence.

(iii) If \(c\) is a real number in the open interval \(I\), and if \(f : I \to \mathbb{R}\) is differentiable at \(c\), then \(f\) is continuous at \(c\).

True. Since \(f\) is differentiable at \(c\), the derivative

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]

exists as a real number. Thus by the product rule for limits,

\[
\lim_{x \to c} [f(x) - f(x)] = \lim_{x \to c} ([f(x) - f(c)] \cdot \frac{x - c}{x - c})
\]

\[
= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c) = f'(c) \cdot 0 = 0.
\]

Thus \(\lim_{x \to c} f(x) = f(c)\) so that \(f\) is continuous at \(c\).
2. (12) Define $s_1 = 1$ and for $n \geq 1$, let $s_{n+1} = \sqrt{3s_n + 1}$.

(i) Prove that $(s_n)$ is an increasing sequence.

We use mathematical induction for the proof. Base case $n = 1$: we know $0 < s_1 = 1 < 2 = \sqrt{4} = \sqrt{3 \cdot s_1 + 1} = s_2$. Thus $0 < s_1 < s_2$. Now for the induction step. Assume $0 < s_k < s_{k+1}$. Then $3 \cdot s_k < 3 \cdot s_{k+1}$. Adding 1 to both sides gives $0 < 3 \cdot s_k + 1 < 3 \cdot s_{k+1} + 1$. Taking square roots gives $0 < \sqrt{3s_k + 1} < \sqrt{3s_{k+1} + 1}$, or $0 < s_{k+1} < s_{k+2}$. By the theory of mathematical induction we obtain $0 < s_n < s_{n+1}$ for all $n \in \mathbb{N}$, so that $(s_n)$ is an increasing sequence of positive numbers.

(ii) It is possible to show $s_n < 4$, $\forall n \in \mathbb{N}$. Assuming this, deduce that $(s_n)$ is a convergent sequence and compute its limit. Be sure to justify your reasoning.

Part (i) has shown us that $(s_n)$ is a strictly increasing sequence of positive numbers. We are allowed to assume $s_n < 4$ for all $n \in \mathbb{N}$. By the Monotone Convergence Theorem for sequences, $\lim_{n \to \infty} s_n$ exists, and $\lim_{n \to \infty} s_n = \sup s_n$. Denote this limit by $s$. We then have

$$s = \lim_{n \to \infty} s_{n+1} = \lim_{n \to \infty} \sqrt{3s_n + 1} = \sqrt{3s + 1}.$$ 

Thus $s = \sqrt{3s + 1}$ so that $s^2 = 3s + 1$ and $s^2 - 3s - 1 = 0$. The quadratic formula gives

$$s = \frac{3 \pm \sqrt{9 + 4}}{2} = \frac{3 \pm \sqrt{13}}{2}.$$ 

We must have $s > 0$ since all $s_n > 0$. Thus $s = \frac{3 + \sqrt{13}}{2}$. 
3. (12)

(i) Suppose that \( f : (a, b) \rightarrow \mathbb{R} \) is continuous and that \( f \) is continuous at \( c \in (a, b) \). Prove that there exists \( M > 0 \) and \( \delta > 0 \) such that

\[
|f(x)| < M \ \forall \ x \in (c - \delta, c + \delta).
\]

Let \( \epsilon = 1 \) in the definition of continuity. Then since \( c \) is an interior point of \((a, b)\), there exists \( \delta > 0 \) such that \((c - \delta, c + \delta) \subseteq (a, b)\) and also such that whenever \( x \in (c - \delta, c + \delta) \),

\[
|f(x) - f(c)| < \epsilon = 1.
\]

Thus whenever \( x \in (c - \delta, c + \delta) \), \( |f(x) - f(c)| < 1 \). But the reverse triangle inequality shows us that \( |f(x)| - |f(c)| \leq |f(x) - f(c)| \) always. Hence, whenever \( x \in (c - \delta, c + \delta) \), \( |f(x)| - |f(c)| \leq |f(x) - f(c)| < 1 \), so that \( |f(x)| < |f(c)| + 1 = M \), whenever \( x \in (c - \delta, c + \delta) \).

(ii) Prove that the equation

\[
3x^3 = 4^x
\]

has a solution in the interval \([1, 2]\). You may assume that \( g(x) = 4^x \) is continuous on \( \mathbb{R} \).

Let \( f : [1, 2] \rightarrow \mathbb{R} \) be defined by \( f(x) = 3x^3 - 4^x \). Then \( f \) is continuous on \([1, 2]\) since it is the difference of two continuous functions there. We note that \( f(1) = 3 \cdot 1 - 4^1 = 3 - 4 = -1 < 0 \), and \( f(2) = 3 \cdot 8 - 4^2 = 24 - 16 = 8 > 0 \). By the Intermediate Value Theorem there exists \( c \in (1, 2) \) such that \( f(c) = 0 \). Then \( 3c^3 - 4^c = 0 \), so that \( 3c^3 = 4^c \), and we have the desired solution to the equation.
4. (13) Consider the function \( f : [0, \frac{1}{\pi}] \rightarrow \mathbb{R} \) defined by \( f(x) = x \cdot \cos \left( \frac{1}{x} \right) \), \( x \neq 0 \), and \( f(0) = 0 \).

(i) Prove that \( f \) is continuous on the closed and bounded interval \([0, \frac{1}{\pi}]\), hence uniformly continuous there. Be sure to justify your reasoning.

The function \( g(x) = x \) is continuous on \( \mathbb{R} \), as is the function \( \cos x \). The function \( h(x) = \frac{1}{x} \) is continuous on the open interval \((0, \infty)\). Since the composition of continuous functions is continuous, \( \cos \circ h(x) = \cos \left( \frac{1}{x} \right) \) is continuous on \((0, \infty)\) and in particular on \((0, \frac{1}{\pi}]\). The product of continuous functions is continuous, so that \( x \cdot \cos \left( \frac{1}{x} \right) \) is also continuous on \((0, \frac{1}{\pi}]\).

For \( x = 0 \) we note that for \( x \neq 0 \), \(|f(x) - f(0)| = |x \cdot \cos \left( \frac{1}{x} \right) - 0| = |x \cos \left( \frac{1}{x} \right)|\). Thus for \( x \neq 0 \),

\[-|x| \cdot 1 \leq f(x) - f(0) = x \cdot \cos \left( \frac{1}{x} \right) \leq |x|.

Since \( \lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0 \), we obtain \( \lim_{x \to 0} [f(x) - f(0)] = 0 \), and \( f \) is continuous at 0.

(ii) Is \( f \) differentiable at \( x = 0 \)? Why or why not? We consider for \( x \neq 0 \) the difference quotient \( \frac{f(x) - f(0)}{x - 0} \).

\[
\frac{f(x) - f(0)}{x - 0} = \frac{x \cos \left( \frac{1}{x} \right) - 0}{x} = \cos \left( \frac{1}{x} \right).
\]

Since \( \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \cos \left( \frac{1}{x} \right) \) does not exist, \( f \) is not differentiable at 0. Proof that \( \lim_{x \to 0} \cos \left( \frac{1}{x} \right) \) does not exist: consider the sequence \((x_n) = (\frac{1}{n\pi})\). It is evident that \( \lim_{n \to \infty} x_n = 0 \). On the other hand, \( \cos \left( \frac{1}{x_n} \right) = \cos (n\pi) = (-1)^n \), which does not have a limit as \( n \to \infty \). It follows by the sequential criterion that \( \lim_{x \to 0} \cos \left( \frac{1}{x} \right) \) does not exist.