1. (13) For each of the following statements, either prove it is true, or provide a counterexample to show that it is false.

(i) For each set A, $\{A\} \in 2^A$.

False. Recall 2^A is the set of all subsets of A. So certainly, $A \in 2^A$, as A is a subset of itself, and $\{A\} \subset 2^A$. But $\{A\}$ is not an element of 2^A. For example, let $A = \{a\}$. Then $2^A = \{\emptyset, \{a\}\}$, and the only two elements of 2^A are $\emptyset$ and $\{a\}$. The set $\{A\} = \{\{a\}\}$ is not an element of A.

(ii) If A and B are non-empty sets and $f : A \to B$ is a function, then $f(f^{-1}(B)) = B$.

False. Let $A = \{a_1, a_2\}$ and let $B = \{b_1, b_2\}$. Define $f : A \to B$ by $f(a_1) = b_1$, $f(a_2) = b_1$. Then $f^{-1}(B) = \{a \in A : f(a) \in B\} = \{a_1, a_2\}$, and $f(\{a_1, a_2\}) = \{f(a_1), f(a_2)\} = \{b_1\}$. So $f(f^{-1}(B)) = \{b_1\} \neq B$.

(iii) If (X, d) is a metric space, then for any $x, y, z \in X$,

$$d(x, y) - d(y, z) \leq d(x, z).$$

True. By condition [M4] in the definition of metric space,

$$d(x, z) \leq d(x, y) + d(y, z), \ \forall x, y, z \in X.$$

Interchange the roles of y and z to get

$$d(x, y) \leq d(x, z) + d(z, y), \ \forall x, y, z \in X.$$

Use rule [M3] to deduce $d(z, y) = d(y, z)$ so that

$$d(x, y) \leq d(x, z) + d(y, z), \ \forall x, y, z \in X.$$

Subtract $d(y, z)$ from both sides to get

$$d(x, y) - d(y, z) \leq d(x, z), \ \forall x, y, z \in X.$$

2. (12)

Let A and B be subsets of the set S. Let $C(A)$ and $C(B)$ denote the complements of A and B in S, respectively. Prove that $C(A) \subset B$ if and only if $A \cup B = S$.

Assume $C(A) \subset B$. Let $x \in A \cup B$. Then $x \in S$, since $A \subset S$ and $B \subset S$. Thus $A \cup B \subset S$. To prove the reverse inclusion, let $x \in S$. Since $A \cup C(A) = S$, either $x \in A$ or $x \in C(A)$. If $x \in A$, then $x \in A \cup B$, since x is in A or B. If $x \in C(A)$, then $x \in B$, since $C(A) \subset B$. Thus $x \in A \cup B$. It follows that $S \subset A \cup B$. Thus the reverse inclusions prove that $A \cup B = S$, assuming that $C(A) \subset B$.

Now assume that $A \cup B = S$. Let $x \in C(A)$. Then $x \in S$, but $x \notin A$. Since $S = A \cup B$, x must be in either A or B. Since we know $x \notin A$, we must have $x \in B$. It follows that $C(A) \subset B$, as we desired to show.

3. (12) Let $E = \{G, L, J\}$. Define a relation R on E by setting

$$R = \{(G,G),(G,J),(L,L),(L,J),(J,J),(J,G),(J,L)\} \subset E \times E.$$

(i) Is R a reflexive relation on E?

Yes. For every $x \in E$, $(x, x) \in R$, since (G,G), (L,L), $(J,J) \in R$. Thus R is reflexive.

(ii) Is R a symmetric relation on E?

Yes. Whenever $(a, b) \in R$, then $(b, a) \in R$. We note that (G, J) and $(J, G) \in R$, and (L, J) and $(J, L) \in R$. Also (G,G), (L,L), $(J,J) \in R$.

(iii) Is R an equivalence relation on E?

No. A relation is an equivalence relation if it is reflexive and symmetric and transitive. We have shown that R is reflexive and symmetric. However, R is not transitive, i.e. if $(a, b) \in R$ and $(b, c) \in R$, it is not necessarily true that $(a, c) \in R$. For example, $(G, J) \in R$, and $(J, L) \in R$, but $(G, L) \notin R$. Thus R is not transitive, so it cannot be an equivalence relation.

4. (13) Define $d_1 : \mathbb{R}^2 \to \mathbb{R}$ by

$$d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

(i) Prove that $(\mathbb{R}^2, d_1)$ is a metric space.

We verify the axioms of a metric space $[M1]$, $[M2]$, $[M3]$, and $[M4]$. For $[M1]$, we note that $d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2| \geq 0$, since absolute values are always non-negative real numbers, and two non-negative real numbers added together have a non-negative sum.

For $[M2]$, we note that if $(x_1, x_2) = (y_1, y_2)$, then $x_1 = y_1$ and $x_2 = y_2$, so that $|x_1 - y_1| = 0 = |x_2 - y_2|$ and $d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2| = 0 + 0 = 0$. On the other hand, if $d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2| = 0$, then we must have $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$ (if either were positive, the sum would be positive) so that $x_1 - y_1 = 0$ and $x_2 - y_2 = 0$, which implies that $x_1 = y_1$ and $x_2 = y_2$. This means that $(x_1, y_1) = (x_2, y_2)$. For $[M3]$, we remark that $|x - y| = 1 \cdot |x - y| = |-1| \cdot |x - y| = |(-1)(x - y)| = |y - x|$, all $x, y \in \mathbb{R}$. It follows that for any (x_1, x_2), $(y_1, y_2) \in \mathbb{R}^2$,

$$d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$
$$= |y_1 - x_1| + |y_2 - x_2| = d_1((y_1, y_2), (x_1, x_2)).$$

For $[M4]$, let (x_1, x_2), (y_1, y_2), $(z_1, z_2) \in \mathbb{R}^2$,

$$d_1((x_1, x_2), (z_1, z_2)) = |x_1 - z_1| + |x_2 - z_2| = |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2|$$
$$\leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2| = |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2|$$
$$= d_1((x_1, x_2), (y_1, y_2)) + d_1((y_1, y_2), (z_1, z_2)).$$

Hence

$$d_1((x_1, x_2), (z_1, z_2)) \leq d_1((x_1, x_2), (y_1, y_2)) + d_1((y_1, y_2), (z_1, z_2)),$$

so Axiom $[M4]$ is satisfied. Therefore $(\mathbb{R}^2, d_1)$ is a metric space.

(ii) In $\mathbb{R}^2$, graph (i.e. draw a picture of) the set

$$\{(x_1, x_2) \in \mathbb{R}^2 : d_1((x_1, x_2), (1, 2)) \leq 1\}.$$

We want to graph the set of all points in $\mathbb{R}^2$ that satisfy

$$|x_1 - 1| + |x_2 - 2| \leq 1.$$

Four cases ensue: $x_1 \geq 1$, and $x_2 \geq 2$; $x_1 < 1$ and $x_2 \geq 2$; $x_1 \geq 1$ and $x_2 < 2$, and $x_1 < 1$ and $x_2 < 2$. When we graph all four possibilities, we get a diamond in $\mathbb{R}^2$, of side length $\sqrt{2}$, centered at $(1, 2)$, including the inside and the boundary.

Name: ____________________

University of Colorado

Mathematics 3200: First Midterm Exam

October 3, 2007

Problem	Points	Score
1	13	
2	12	
3	12	
4	13	
Total	50	