

1. (12) For each of the following series, determine whether it is absolutely convergent, conditionally convergent, or divergent. Be sure to explain your reasoning.

(i)

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n + 1}{3^n - \cos n};$$

**Solution:**

$$\left| (-1)^n \frac{2^n + 1}{3^n - \cos n} \right| = \frac{2^n + 1}{3^n - \cos n} \leq \frac{2^n + 1}{3^n - 1}.$$

Use the limit comparison test:

$$\lim_{n \rightarrow \infty} \frac{(2^n + 1)/(3^n - 1)}{2^n/3^n} = \lim_{n \rightarrow \infty} \frac{(1 + \frac{1}{2^n})}{(1 - \frac{1}{3^n})} = 1.$$

Since  $\sum_{n \rightarrow \infty} \frac{2^n}{3^n}$  converges (a geometric series with ratio  $\frac{2}{3} < 1$ ), the limit comparison test shows that  $\sum_{n=1}^{\infty} \frac{2^n + 1}{3^n - 1}$  converges, and then the regular comparison test shows that  $\sum_{n=1}^{\infty} \frac{2^n + 1}{3^n - \cos n}$  converges. Thus  $\sum_{n=1}^{\infty} (-1)^n \frac{2^n + 1}{3^n - \cos n}$  converges absolutely.

(ii)

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{(n \log(n))^a}, \quad a > 1;$$

**Solution:** Note that for  $n \geq 3$ ,  $\log(n) > 1$  so that  $\frac{1}{\log(n)} < 1$ . Thus for  $n \geq 3$ ,  $\left| (-1)^n \frac{1}{(n \log(n))^a} \right| = \frac{1}{n^a} \cdot \frac{1}{[\log(n)]^a} < \frac{1}{n^a}$ . Since  $a > 1$ , we know that  $\sum_{n=1}^{\infty} \frac{1}{n^a}$  converges, so by the comparison test,  $\sum_{n=1}^{\infty} \frac{1}{(n \log(n))^a}$  converges, thus  $\sum_{n=1}^{\infty} (-1)^n \frac{1}{(n \log(n))^a}$  converges absolutely.

(iii)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{(n+1)^n}.$$

Here

$$\lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} = \lim_{n \rightarrow \infty} \left[ \frac{n}{n+1} \right]^n = \lim_{n \rightarrow \infty} \frac{1}{(1 + \frac{1}{n})^n} = \frac{1}{e} \neq 0.$$

Thus the terms of the series do not go to zero so that  $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{(n+1)^n}$  diverges.

2. (13)

- (i) Suppose that  $\sum_{n=1}^{\infty} a_n$  is convergent series of real numbers. Define  $b_n = a_{2n-1}$  and  $c_n = a_{2n}$ . Is it true that both  $\sum_{n=1}^{\infty} b_n$  and  $\sum_{n=1}^{\infty} c_n$  are convergent? If so, prove this is the case. If not, provide a counterexample.

**Solution:** It is untrue that  $\sum_{n=1}^{\infty} b_n$  and  $\sum_{n=1}^{\infty} c_n$  are convergent. Consider  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  which converges. Then  $b_n = \frac{-1}{2n-1}$  and  $\sum_{n=1}^{\infty} \frac{-1}{2n-1}$  diverges to  $-\infty$ , and  $c_n = \frac{1}{2n}$  so that  $\sum_{n=1}^{\infty} \frac{1}{2n}$  diverges to  $+\infty$ .

- (ii) Fix  $b > 1$ . Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{1+x^n}$$

is uniformly convergent on  $[b, \infty)$ .

**Solution:** Note that since  $b > 1$ ,  $\frac{1}{b} < 1$ . Thus if  $x \geq b > 1$ ,  $x^n \geq b^n$ , and

$$1 + x^n \geq 1 + b^n > b^n > 1.$$

Hence

$$\frac{1}{1+x^n} \leq \frac{1}{1+b^n} \leq \frac{1}{b^n} = \left[\frac{1}{b}\right]^n.$$

Let  $M_n = \left[\frac{1}{b}\right]^n$ . Since  $\frac{1}{b} < 1$ ,  $\sum_{n=1}^{\infty} M_n < \infty$ . Since  $\frac{1}{1+x^n} \leq M_n$  for every  $x \in [b, \infty)$ . By the Weierstrass  $M$ -test,  $\sum_{n=1}^{\infty} \frac{1}{1+x^n}$  is uniformly convergent on  $[b, \infty)$ .

3. (12) Consider the function  $f : [0, 1) \rightarrow \mathbb{R}$  defined by  $f(x) = \frac{1}{1-x}$ . Let  $f_n : [0, 1) \rightarrow \mathbb{R}$  be given for  $n \geq 0$  by  $f_n(x) = \sum_{k=0}^n x^k$ . Prove that  $\{f_n\}$  converges to  $f$  pointwise on  $[0, 1)$ . Is this convergence uniform on  $[0, 1)$ ? Be sure to justify your answer.

**Solution:** Fix  $x \in [0, 1)$ . Note that for each  $n \in \mathbb{N}$ ,  $f_n(x) = \sum_{k=0}^n x^k$  is the  $n$ -th partial sum of the infinite series  $\sum_{n=0}^{\infty} x^n$ . This is a geometric series with ratio  $x \in [0, 1)$  so converges to  $f(x) = \frac{1}{1-x}$ . We now show the convergence is not uniform. Note that we can rewrite  $f_n(x)$  as  $\sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x}$ . Then:

$$|f_n(x) - f(x)| = \left| \frac{1-x^{n+1}}{1-x} - \frac{1}{1-x} \right| = \left| \frac{-x^{n+1}}{1-x} \right| = \frac{x^{n+1}}{1-x}.$$

Now for any fixed  $n \in \mathbb{N}$ ,  $\sup_{x \in [0, 1)} \frac{x^{n+1}}{1-x} = +\infty$ , as the function  $\frac{x^{n+1}}{1-x}$  is strictly increasing on  $[0, 1)$  to  $+\infty$ . Thus

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, 1)} |f_n(x) - f(x)| = \lim_{n \rightarrow \infty} \sup_{x \in [0, 1)} \frac{x^{n+1}}{1-x} = +\infty.$$

Since  $\lim_{n \rightarrow \infty} \sup_{x \in [0, 1)} |f_n(x) - f(x)| \neq 0$ , it follows that the sequence  $\{f_n\}$  does not converge uniformly to  $f$ .

4. (13) Suppose that  $\{f_n : S \rightarrow \mathbb{R}\}_{n=1}^{\infty}$  is a sequence of functions defined on the interval  $S \subset \mathbb{R}$ , with each  $f_n$  bounded on  $S$  by  $M_n > 0$ . Suppose also that the sequence  $\{f_n\}_{n=1}^{\infty}$  converges uniformly on  $S$  to a limit function  $f : S \rightarrow \mathbb{R}$ . Prove that  $f$  is bounded on  $S$ .

**Solution:** Recall since  $\{f_n\}$  converges uniformly on  $S$  to  $f$ , given  $\varepsilon = 1$ , there exists  $N \in \mathbb{N}$  such that for all  $n > N$ , and every  $x \in S$ ,  $|f_n(x) - f(x)| < \varepsilon = 1$ . Let  $n = N + 1 > N$ . Then for all  $x \in S$ ,

$$|f(x)| - |f_{N+1}(x)| \leq |f(x) - f_{N+1}(x)| = |f_{N+1}(x) - f(x)| < 1.$$

Thus  $|f(x)| < 1 + |f_{N+1}(x)|$ , for all  $x \in S$ . Since  $f_{N+1}$  is bounded, we know that  $|f_{N+1}(x)| \leq M_{N+1}$ , for all  $x \in S$ . It follows that for all  $x \in S$ ,

$$|f(x)| < 1 + M_{N+1},$$

and the limit function  $f$  is bounded by  $M_{N+1} + 1$ .