1. (17)

(i) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n e^{1-n^2}$ is absolutely convergent, conditionally convergent, or divergent.

(ii) Determine whether the series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{\log(1/n)}$ is absolutely convergent, conditionally convergent, or divergent.

(iii) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers. If $\sum_{n=1}^{\infty} a_n$ converges, prove that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

2. (17)

(i) Calculate the radius of convergence r for the power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$. Be sure to explain your reasoning.

(ii) Does the power series in part (i) above converge at $x = r$? Does it converge at $x = -r$? Explain. (Here $r > 0$ is the radius of convergence computed in part (i).)

(iii) Prove that the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{1+x^{2n}}$ is uniformly convergent on $[\delta, \infty)$ for any fixed $\delta > 1$. Does the series converge uniformly on $[1, \infty)$?

3. (16)

(i) Prove that for every $x \in \mathbb{R}$ we have

$$\frac{d}{dx}\left[\sum_{n=1}^{\infty} \frac{1}{n^3(1+nx^4)}\right] = \sum_{n=1}^{\infty} \frac{-4x^3}{n^2(1+nx^4)^2}.$$

(ii) Prove that

$$\int_0^1 \frac{\log(1+x)}{x}\,dx = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}.$$

Be sure to justify your reasoning.

4. (17)

(i) Let U be a non-empty open subset of $\mathbb{R}^n$. Suppose $f : U \to \mathbb{R}^m$ is a function defined on U taking on values in $\mathbb{R}^m$. What does it mean to say that f is differentiable at $\vec{a} \in U$?

(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Using the definition, find the directional derivative $D_{\vec{v}}(f)(0, 0)$ for $\vec{v} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(iii) Let f be the function defined in part (ii). Is f differentiable at (0,0)? Explain why or why not.

5. (16)

(i) Let $\vec{a} \in \mathbb{R}^n$, fix $\delta > 0$, and let $U = B(\vec{a}, \delta) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{a}\| < \delta\}$. Suppose $f : U \to \mathbb{R}$ is a differentiable function. Prove that if $\vec{b} \in U$,

$$f(\vec{b}) - f(\vec{a}) = \operatorname{grad}(f)(\vec{x_0}) \cdot (\vec{b} - \vec{a}),$$

for some point $\vec{x_0}$ on the line segment joining $\vec{a}$ and $\vec{b}$. Deduce that if there exists $M > 0$ such that $\|\operatorname{grad}(f)(\vec{x})\| \leq M$ for all $\vec{x} \in U$, then

$$|f(\vec{b}) - f(\vec{a})| \leq M\|\vec{b} - \vec{a}\|.$$

(ii) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Suppose that

$$f(t\vec{x}) = tf(\vec{x}),\ \forall t \in \mathbb{R},\ \forall\, \vec{x} \in\ \mathbb{R}^n.$$

Show that for all $\vec{x} \in\ \mathbb{R}^n$,

$$f(\vec{x}) \ = \ \mathrm{grad}(f)(\vec{0}) \cdot \vec{x}.$$

6. (17)

(i) Let $\vec{a} \in \mathbb{R}^n$, fix $\delta > 0$, and let $U = B(\vec{a}, \delta)$ be defined as in Question 5. Suppose that $f : U \to \mathbb{R}$ is twice continuously differentiable in U. Let $\vec{h} \in \mathbb{R}^n$ with $\|\vec{h}\| < \delta$. Prove that there are continuous functions $\{g_{i,j} : U \to \mathbb{R}\}_{i,j=1}^n$ such that $f(\vec{a} + \vec{h}) = f(\vec{a}) + \operatorname{grad}(f)(\vec{a}) \cdot \vec{h} + \sum_{i,j=1}^n g_{i,j}(\vec{a} + \vec{h}) h_i h_j$. State any theorems you are using.

(ii) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x^4 + y^3 - 2x^2 + 3y^2 + 5.$$

By using part (i), or otherwise, prove that f attains a relative maximum value at $(x, y) = (0, -2)$.

(iii) Use the Implicit Function Theorem to show that the equations

$$x_1y_2 + x_2y_1 + x_3 + y_1^2 = 0,$$

$$x_1x_2x_3 + y_1 + y_2^3 + 1 = 0$$

determine y_1 and y_2 implicitly as continuously differentiable functions of the independent variables x_1, x_2, and x_3 in some open set containing the point $(x_1, x_2, x_3, y_1, y_2) = (1, 1, -1, 1, -1)$. Calculate $\frac{\partial y_1}{\partial x_1}$ and $\frac{\partial y_2}{\partial x_2}$ for $(x_1, x_2, x_3) = (1, 1, -1)$.

Name: ______________________________

University of Colorado

Mathematics 4001: Final Exam

Monday, May 3, 2010

Problem	Points	Score
1	17	
2	17	
3	16	
4	17	
5	16	
6	17	
Total	100	