1. (13) For each of the following statements, either prove it is true, or provide a counterexample to show that it is false.

(i) If (s_n) is a sequence such that $\lim\ \sup s_n = 0$, then $\lim\ \sup |s_n| = 0$.
False. Let $s_n = 0$ for n even, and $s_n = -5$ for n odd. Then $|s_n| = 0$ for n even, and $|s_n| = 5$ for n odd. We see that $\lim\ \sup s_n = 0$, and $\lim\ \sup |s_n| = 0$.

(ii) If a sequence (s_n) in $\mathbb{R}$ is monotone, then every one of its subsequences is monotone.
True. Suppose that (s_n) is non-decreasing. Then $s_n \leq s_{n+1}$ for all n so that $s_n \leq s_m$ whenever $n \leq m$. Let (s_{n_k}) be a subsequence of (s_n). Then for $k < k+1$, $n_k < n_{k+1}$, and it follows that $s_{n_k} \leq s_{n_{k+1}}$ for all $k \in \mathbb{N}$. Thus (s_{n_k}) is non-decreasing. The proof when (s_n) is non-decreasing is similar.

(iii) If I is an interval, and $f : I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.
False. Let $I = (0, \infty)$ and define f on I by $f(x) = \frac{1}{x}$. Then f is continuous on I but it is not uniformly continuous. Note $(\frac{1}{n})$ is a Cauchy sequence in I but $(f(\frac{1}{n}) = (n)$ is not a Cauchy sequence in $\mathbb{R}$.

2. (12) Let (s_n) be the sequence defined by

$$s_1 = 1,$$

and

$$s_{n+1} - s_n = \frac{1}{3^n},\ n \geq 1.$$

Prove that (s_n) is a Cauchy sequence, and compute its imit.
Let $\epsilon > 0$ be given. Find N such that $n > N$ implies

$$\frac{1}{3^{N-1}} \cdot \frac{1}{2} < \epsilon.$$

We now show that if $n > m > N$, $|s_n - s_m| < \epsilon$. This will show that (s_n) is a Cauchy sequence. For $n > m > N$,

$$|s_n - s_m| = |s_n - s_{n-1} + s_{n-1} - s_{n-2} + s_{n-2} - \cdots - s_{m+1} + s_{m+1} - s_m|$$

$$\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \cdots + |s_{m+1} - s_m| = \sum_{k=m}^{n-1} \frac{1}{3^k} = \frac{1}{3^m} \sum_{k=0}^{n-1-m} \frac{1}{3^k}$$

$$< \frac{1}{3^m} \sum_{k=0} \infty \frac{1}{3^k} = \frac{1}{3^m}\frac{3}{2} < \frac{1}{3^{N-1}} \cdot \frac{1}{2} < \epsilon.$$

Thus (s_n) is a Cauchy sequence.

We now prove by induction that $s_n = \sum_{k=0}^{n-1} \frac{1}{3^k}$ for all $n \in \mathbb{N}$. The statement is true for $n = 1$, since $s_1 = 1 = \frac{1}{3^0}$. Assume the statement is true for n; assuming this, we prove the statement is true for $n+1$. Then $s_{n+1} - s_n = \frac{1}{3^n}$ so that

$$s_{n+1} = s_n + \frac{1}{3^n} = \sum_{k=0}^{n-1} \frac{1}{3^k} + \frac{1}{3^n} = \sum_{k=0}^{n+1-1} \frac{1}{3^k}.$$

We have proved the inductive step so that the statement is true for all n. Therefore the sequence (s_n) are the partial sums for the geometric series $\sum_{k=0}^{\infty} \frac{1}{3^k}$. This series sums to $\frac{1}{1-\frac{1}{3}} = \frac{3}{2}$. Thus $\lim_{n\to\infty} s_n = \frac{3}{2}$.

3. (13)

(i) Suppose $f : [a, b] \to \mathbb{R}$ and that f is continous at $c \in (a, b)$, with $f(c) > 0$. Prove that there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$.

Use the definition of continuity, with $\epsilon = \frac{f(c)}{2} > 0$. Then there exists $\delta > 0$ such that if $x \in (c - \delta, c + \delta)$ $|f(x) - f(c)| < \frac{f(c)}{2}$. That is, if $x \in (c - \delta, c + \delta)$,

$$-\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2}$$

so that

$$\frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}$$

whenever $x \in (c - \delta, c + \delta)$. Since $f(c) > 0$, it follows that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$.

(ii) Prove that the equation $3x = e^x$ has a solution in the interval $[0, 1]$. (You may use the fact that the approximation of e valid to four digits is 2.71828.)

Let $h : [0, 1] \to \mathbb{R}$ be defined by $h(x) = 3x - e^x$. Then h is continuous since the functions $f(x) = 3x$ and $g(x) = e^x$ are continuous on $[0, 1]$. We note that $h(0) = 3 \cdot 0 - e^0 = 0 - 1 = -1 < 0$. Also $h(1) = 3 - e^1 = 3 - e \sim 3 - 2.71828 > 0$. By the Intermediate Value Theorem, there exists $x_0 \in (0, 1)$ with $h(x_0) = 0$. Thus $3x_0 - e^{x_0} = 0$ so that

$$3x_0 = e^{x_0}.$$

4. (12)

(i) Prove that the function $f : [0, \frac{1}{\pi}] \to \mathbb{R}$ defined by $f(x) = x \sin\frac{1}{x},\ x \neq 0$ and $f(0) = 0$, is continuous on $[0, \frac{1}{\pi}]$. Be sure to justify your reasoning.
The function $g(x) = \frac{1}{x}$ is continuous on $(0, \frac{1}{\pi}]$. The function $h(x) = \sin x$ is continuous on $[\pi, \infty)$. Therefore, the composed function $h \circ g(x) = \sin\frac{1}{x}$ is continous on $(0, \frac{1}{\pi}]$. The product of two continuous functions is continuous, so it follows that the function $x \sin\frac{1}{x}$ is continous on $(0, \frac{1}{\pi}]$. At the point $x = 0$, we note that

$$|x \cdot \sin\frac{1}{x} - f(0)| = |x \cdot \sin\frac{1}{x} - 0| = |x| \cdot |\sin\frac{1}{x}| \leq |x| \cdot 1 = |x|.$$

Thus

$$-|x| \ \leq x \cdot \sin\frac{1}{x} - f(0) \leq |x|,\ x \in (0, \frac{1}{\pi}].$$

It follows that for any sequence (x_n) tending to 0 in $(0, \frac{1}{\pi}]$,

$$-|x_n| \ \leq x_n \cdot \sin\frac{1}{x_n} - f(0) \leq |x_n|,$$

so that

$$\lim x_n \cdot \sin\frac{1}{x_n} \ = \ 0 \ = \ f(0)$$

by the Squeeze Theorem. Thus f is continous at 0 also.

(ii) Is the function f is part (i) above uniformly continuous on $[0, \frac{1}{\pi}]$? Justify your reasoning.
The function f is continuous on the closed and bounded interval $[0, \frac{1}{\pi}]$. By a fundamental theorem on uniform continuity, it follows that f is uniformly continuous on $[0, \frac{1}{\pi}]$.

Name: ______________________________

University of Colorado

Mathematics 4310: Second Midterm Exam

November 5, 2007

Problem	Points	Score
1	13	
2	12	
3	13	
4	12	
Total	50	