1. (25) For each of the following statements, about either prove it is true, or provide a counterexample to show that it is false.

(i) If S is a non-empty bounded subset of $\mathbb{R}$ and $c \in \mathbb{R}$, let $c\,S = \{cs : \; s \in S\}$. Then

$$\sup c\,S \;=\; c \sup S.$$

(ii) If a sequence is convergent, every one of its subsequences is convergent.

(iii) If a sequence of bounded functions (f_n) converges uniformly on $S \subset \mathbb{R}$ to the limit function f, then f is bounded on S.

2. (25)

(i) Define a sequence (x_n) by $x_1 = 1$ and $x_{n+1} = \sqrt{3 + 2x_n}$, $n \geq 1$. By using an appropriate convergence theorem or otherwise, prove that (x_n) is convergent, and calculate $\lim x_n$.

(ii) Let (a_n) be a sequence of real numbers. If (a_n) is bounded, prove that the power series $\sum_{n=1}^{\infty} \frac{a_n}{n^2}\, x^n$ converges uniformly on $[-1, 1]$ to a continuous function.

3. (25)

(i) Let $f(x) \geq g(x) \geq h(x)$ for all $x \in (a,b)$, and fix $c \in (a,b)$. If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L \in \mathbb{R}$, prove that $\lim_{x\to c} g(x) = L$.

(ii) Is the function $f(x) = x\cos\frac{1}{x}$ uniformly continuous on $(0, \frac{1}{\pi})$? Justify your answer.

4. (25)

(i) Give an example of a sequence of functions (f_n) defined on $S \subseteq R$ that converges pointwise but not uniformly to the limit function $f : S \to \mathbb{R}$. Be sure to explain why the sequence does not converge uniformly.

(ii) Let (f_n) be a sequence of continuous functions that converges uniformly to the limit function f on the interval $[a, b]$. Suppose that the sequence $(x_n) \subseteq [a, b]$ converges to $x_0 \in [a, b]$. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x_0).$$

5. (25)

(i) Let $p(x) = 7x^8 - 8x^7 + C$, where C is a fixed constant. Prove that there exists at most one value $r \in [1, \infty)$ such that $p(r) = 0$.

(ii) Consider the sequence $(y_n) = (\int_2^4 \frac{1}{1+x^n} dx)$. Compute with justification

$$\lim_{n \to \infty} y_n.$$

(iii) Recall that the hyperbolic cosine function is on $\mathbb{R}$ by $\cosh x = \frac{e^x + e^{-x}}{2}$, $x \in \mathbb{R}$. Find the Taylor series for $\cosh x$ and prove it converges to $\cosh x$ for all $x \in \mathbb{R}$.

6. (25)

(i) Consider the function f defined on the interval $[0,2]$ by

$$f(x) = \begin{cases} 0 & x \text{ rational}, \\ x & x \text{ irrational}. \end{cases}$$

Compute the lower Darboux integral $L(f)$ and the upper Darboux integral $U(f)$ on the interval $[0,2]$, and determine whether or not f is integrable on $[0,2]$.

(ii) Let $f : [0,1] \to \mathbb{R}$ be a bounded function. Suppose there exist two sequences of partitions $\{\mathcal{P}_n\}_{n=1}^{\infty}$ and $\{\mathcal{Q}_n\}_{n=1}^{\infty}$ of $[0,1]$ such that

$$\lim_{n\to\infty} [U(f, \mathcal{Q}_n) - L(f, \mathcal{P}_n)] \;=\; 0.$$

Prove that f is integrable over $[0,1]$.