

For the purposes of this exam, you might want to use the integral formulas

$$\int e^{\alpha x} dx = \frac{e^{\alpha x}}{\alpha} + C, \quad \alpha \in \mathbb{C}, \quad \alpha \neq 0,$$

$$\int x^2 \cos nx dx = \frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} + C,$$

$$\int x^4 \cos nx dx = \frac{x^4 \sin nx}{n} + \frac{4x^3 \cos nx}{n^2} - \frac{12x^2 \sin nx}{n^3} - \frac{24x \cos nx}{n^4} + \frac{24 \sin nx}{n^5} + C.$$

1. (15) Let $f(x) = e^x$, $-\pi < x \leq \pi$. Consider the 2π -periodic extension of f to a function defined on all of \mathbb{R} .

(i) Prove that the exponential form of the Fourier series for f is given by

$$f(x) \sim \frac{e^{\pi} - e^{-\pi}}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(1 - in)} e^{inx}.$$

The exponential Fourier coefficients are given by

$$\begin{aligned} c_n(f) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-in)x} dx \\ &= \frac{1}{2\pi} \left[\frac{e^{(1-in)x}}{1-in} \right]_{x=-\pi}^{\pi} = \frac{1}{2\pi} \left[\frac{e^{(1-in)\pi} - e^{(1-in)(-\pi)}}{1-in} \right] = \frac{e^{\pi} - e^{-\pi}}{2\pi(1-in)} \cdot (-1)^n. \end{aligned}$$

Here we use that $e^{in\pi} = e^{-in\pi} = (-1)^n$. Thus the Fourier series is given by

$$f(x) \sim \frac{e^{\pi} - e^{-\pi}}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(1 - in)} e^{inx}.$$

(ii) What is the value of this series at $x = \pi$? Justify your answer.

We note $f'(x) = e^x$ for $-\pi < x < \pi$, does not exist at π or $-\pi$, and can be extended periodically except at odd multiples of π . Therefore f is piecewise smooth so that by Dirichlet's Theorem, the value of the series at π is equal to

$$\frac{f(\pi-) + f(\pi+)}{2} = \frac{e^{\pi} + e^{-\pi}}{2}.$$

2. (15)

(i) Let f be the 2π -periodic function defined on $(-\pi, \pi]$ by $f(x) = x^4 - 2\pi^2 x^2$.Prove that the trigonometric form of the Fourier series for f is given by

$$f(x) \sim -\frac{7}{15}\pi^4 + 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos nx.$$

We note f is even so that $b_n(f) = 0$ all $n \geq 1$ and $a_n(f) = \frac{2}{\pi} \int_0^{\pi} [x^4 - 2\pi^2 x^2] dx$.So for n not equal to 0,

$$\begin{aligned} a_n(f) &= \frac{2}{\pi} \left[\frac{x^4 \sin nx}{n} + \frac{4x^3 \cos nx}{n^2} - \frac{12x^2 \sin nx}{n^3} - \frac{24x \cos nx}{n^4} + \frac{24 \sin nx}{n^5} \right]_0^{\pi} \\ &+ \frac{2}{\pi} \cdot (-2\pi^2) \left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{\pi} = \frac{2}{\pi} \left[\frac{4\pi^3 (-1)^n}{n^2} - \frac{24\pi (-1)^n}{n^4} \right] - 4\pi \left[\frac{2\pi (-1)^n}{n^2} \right] \\ &= \frac{8\pi^2 (-1)^n}{n^2} + \frac{48(-1)^{n+1}}{n^4} - \frac{8\pi^2 (-1)^n}{n^2} = \frac{48(-1)^{n+1}}{n^4}. \end{aligned}$$

$$a_0(f) = \frac{2}{\pi} \int_0^{\pi} [x^4 - 2\pi^2 x^2] dx = \frac{2}{\pi} \left[\frac{x^5}{5} - \frac{2\pi^2 x^3}{3} \right]_{x=0}^{\pi} = \frac{2\pi^4}{5} - \frac{4\pi^4}{3} = -\frac{14}{15}.$$

We therefore obtain $f(x) \sim -\frac{7}{15}\pi^4 + 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos nx$.(ii) Using part (i) and the fact that the fact that $\cos n\pi = (-1)^n$ for all integers n , or otherwise, find a formula in terms of π^4 for

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Note that f is piecewise smooth and is in fact continuous at $x = \pi$ since $f(\pi-) = -\pi^4 = f(\pi+) = f(\pi)$. So by Dirichlet's Theorem, setting $x = \pi$ we have

$$-\pi^4 = -\frac{7}{15}\pi^4 + 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos n\pi$$

so

$$-\pi^4 = -\frac{7}{15}\pi^4 - 48 \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Solving, we get $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. (This is a famous formula of Euler.)

3. (15) Consider the 2-periodic function g defined on $(-1, 1]$ by

$$g(x) = -2x - 1, -1 < x \leq 0, \quad g(x) = 2x - 1, 0 < x \leq 1.$$

(i) Draw a graph of this function over the interval $(-3, 3]$.

This graph is periodic, with period 2, so that after you draw something that looks like $y = |2x|$ shifted down one unit (“V”-shape) on the interval $(-1, 1]$, you need to repeat the pattern so the graph on $(-3, 3]$ looks like 3 “V”’s strung together.

(ii) Is the 2-periodic extension of g to all of \mathbb{R} an odd function, an even function, or neither? Justify your answer.

By observation of the graph, the function g is even, i.e. its graph is symmetric with respect to the y -axis. Also, for $x \in [0, 1]$, we have $-x \in [-1, 0]$ so that $g(-x) = -2(-x) + 1 = 2x + 1 = g(x)$.

(iii) Suppose you have computed the Fourier series for the 2-periodic function g in terms of the functions $\{\cos \pi nx : n \geq 0\}$ and $\{\sin \pi nx : n \geq 1\}$. What is the value of this series at $x = 1$? Justify your answer.

Note that $g'(x) = -2$ for $x \in (-1, 0)$ and $g'(x) = 2$ for $x \in (0, 1)$. The derivative of g does not exist at -1 , 0 , or 1 . Extending this 2-periodically, we see that g is piecewise smooth. Also, we see that from the graph of g that g is continuous at every point. By the Corollary to Dirichlet’s Theorem, value of the Fourier series for g evaluated at $x = 1$ is equal to $g(1) = 2 - 1 = 1$.

4. (15) Let $f(x) = x$, $-\pi < x \leq \pi$.

(i) Recall that the trigonometric form of the Fourier series for f is given by:

$$f(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Use this formula to prove that for $g(x) = x$, $-3 < x < 3$, we have

$$g(x) = \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{\pi nx}{3}, \quad -3 < x < 3.$$

We note that f is piecewise smooth and continuous on $(-\pi, \pi)$. So for every $y \in (-\pi, \pi)$, by Dirichlet's Theorem we obtain

$$f(y) = y = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin ny.$$

Now put $y = \frac{\pi x}{3}$. Whenever $-3 < x < 3$ we have $-\pi < \frac{\pi x}{3} < \pi$, so that

$$\frac{\pi x}{3} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{3},$$

so that for $x \in (-3, 3)$, we have

$$x = \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{\pi nx}{3}.$$

(ii) Suppose we define $g(3) = 0$ and extend g to a 6-periodic function defined on all of \mathbb{R} . Determine whether the following statement is true or false. Justify your answer.

“Let $S_N^g(x) = \frac{6}{\pi} \sum_{n=1}^N \frac{(-1)^{n+1}}{n} \sin \frac{\pi nx}{3}$. Then $S_N^g(x)$ converges to $g(x)$ **uniformly** on the interval $(-3, 3]$.”

The function g as defined above will be 6-periodic, piecewise smooth, and averaged. Therefore it is true that for every x in \mathbb{R} , the partial sums $S_N^g(x)$ will converge to $g(x)$ **pointwise**. However, the interval $(-3, 3]$ contains 3 and g is not continuous at 3. Thus Gibbs' phenomenon holds at $x = 3$ and in fact it is possible to show that there is a sequence $\{x_N\}$ in the interval $(-3, 3]$ converging to 3 with $\frac{S_N^g(x_N) - g(x_N)}{6} > 0.089$. So we cannot have uniform convergence, since that requires

$\lim_{N \rightarrow \infty} \sup_{x \in \mathbb{R}} |S_N^g(x) - g(x)| = 0$, which evidently does not occur.