

1. (17) Let $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$, $-\pi < x \leq \pi$. Consider the 2π -periodic extension of f to a function defined on all of \mathbb{R} .

(i) Prove that the trigonometric form of the Fourier series for f is given by

$$f(x) \sim \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} \cos(nx).$$

(ii) State, without proof, the main pointwise convergence theorem for Fourier series for piecewise smooth functions with period 2π (at a point $x_0 \in \mathbb{R}$).

(iii) By using parts (i) and (ii) above or otherwise, find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}.$$

(iv) Using the formula in (i) above, compute directly the trigonometric form of the Fourier series for $g(x) = \sinh x$. Justify your methods.

2. (16) Consider the sequence of functions f_N ($N = 3, 4, 5, \dots$) defined on $[0, 1]$ by

$$f_N(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{N}, \\ \sqrt{N}, & \frac{1}{N} \leq x < \frac{2}{N}, \\ 0, & \frac{2}{N} \leq x \leq 1. \end{cases}$$

(i) Show that the sequence of functions $\{f_N\}_{N=3}^{\infty}$ converges pointwise to the function $f(x) = 0$ on $[0, 1]$.

Determine whether each of the following statements is true or false. Be sure to justify your reasoning. The sequence of functions $\{f_N\}_{N=3}^{\infty}$ referred to is the same as in part (i).

(ii) The sequence $\{f_N\}$ converges uniformly to $f(x) = 0$ on $[0, 1]$.

(iii) The sequence $\{f_N\}$ converges to $f(x) = 0$ in the mean square norm on $L^2[0, 1]$.

3. (17) Consider the functions ϕ_1 , ϕ_2 and ϕ_3 in $L^2[0, 1]$ defined by

$$\phi_1(x) = \sqrt{2}\chi_{[\frac{1}{2}, 1)}, \quad \phi_2(x) = \sqrt{6}\chi_{[\frac{1}{3}, \frac{1}{2})}, \quad \phi_3(x) = \sqrt{12}\chi_{[\frac{1}{4}, \frac{1}{3})}.$$

(i) Show that $\{\phi_1, \phi_2, \phi_3\}$ is an orthogonal set in $L^2[0, 1]$ and calculate the norms $\|\phi_j\|$, $j = 1, 2, 3$.

(ii) Among all possible linear combinations of the form

$$\sum_{j=1}^3 d_j \phi_j,$$

find the coefficients that give the best approximation in the mean square norm to the function

$$f(x) = 3x$$

in $L^2[0, 1]$.

(iii) For the choice of $\{d_j\}_{j=1}^3$ found in part (ii), is $\sum_{j=1}^3 d_j \phi_j(x) = 3x$ for almost all $x \in [0, 1]$? Justify your answer.

4. (16)

- (i) Let f and g be functions in $L^1(\mathbb{R})$. Defining $f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$, verify that

$$f * g(x) = g * f(x) \text{ almost everywhere.}$$

- (ii) Let $f(x) = x^2 + 2$, $g(x) = 3x\chi_{[-2,1]}(x)$. Compute $f * g(x)$ for all $x \in \mathbb{R}$. Is $f * g$ an even function, an odd function, or neither?

5. (17)

(i) Let $f(x) = e^{-2\pi|x|}$. Using the definition of Fourier transform, find $\widehat{f}(s)$.

(ii) Using part (i) or otherwise, compute the Fourier transform for the function k defined by

$$k(x) = e^{4\pi ix} e^{-2\pi|x-\frac{1}{8}|}.$$

(iii)

Let $h(x) = [\text{sinc}(\pi x)]^2$. Prove, with justification, that

$$\widehat{h}(s) = (1 - |s|)\chi_{[-1,1]}(s).$$

[Hint: you may first want to write down a formula for $\chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]}(s)$.]

6. (17)

(i) Let

$$f(x) = e^{-6\pi ix} \operatorname{sinc}(4\pi(x-3)).$$

Calculate $\widehat{f}(s)$. Be sure to justify your steps.

(ii) Using the Plancherel Theorem or otherwise, calculate $\int_{-\infty}^{\infty} |f(x)|^2 dx$, where f is as in part (i).

(iii) Let $g(x) = 2x$, $N = 4$, and $l = 1$. Compute the discrete Fourier transform \tilde{g} of g corresponding to these values of N and l . Calculate $|\tilde{g}_0 - c_0(g)|$, where $c_0(g)$ is the 0^{th} exponential Fourier coefficient for g in $L^2[0, 1]$ with respect to the orthonormal basis $\{e^{2\pi i n x} : n \in \mathbb{Z}\}$.

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Problem	Points	Score
1	17	
2	16	
3	17	
4	16	
5	17	
6	17	
Total	100	