A survey of projective multi-resolution analyses and a projective multi-resolution analysis corresponding to the quincunx lattice

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Dedicated to Larry Baggett, colleague, mentor and friend.

Summary. We give a survey of the concept of projective multi-resolution analyses as introduced by M. Rieffel and studied further by M. Rieffel and the author. We give examples of projective multi-resolution analyses corresponding to the non-diagonal 2×2 integer dilation matrix $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ that has determinant -2, and also to the non-diagonal 2×2 matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ having determinant 2 related to the quincunx lattice. The method of construction follows that given by Rieffel and the author in their earlier work, but also poses new problems. In both examples given here, the one-dimensional initial $C(\mathbb{T}^2)$-modules are not free, but the in the quincunx case, the one-dimensional wavelet module is free, whereas in the case corresponding to the dilation matrix whose determinant is negative, the one-dimensional wavelet module is not free either.

1 Introduction

Let A be an $n \times n$ integer dilation matrix. In previous papers ([20], [17]), M. Rieffel and the author, and then the author alone, studied the existence of projective multi-resolution analyses in Ξ corresponding to dilation by certain $n \times n$ matrices A. In [20], the general abstract construction was given, and the examples of cases where A was a 2×2 diagonal matrix was emphasized, and in [17], examples of projective multi-resolution analyses corresponding to diagonal dilation matrices with $n \geq 3$ were studied in detail. A very important module used in both [20] and [17] was the module Ξ, where we recall that for $n \in \mathbb{N}$, the right $C(\mathbb{T}^n)$ module Ξ is defined as the completion of $C_c(\mathbb{R}^n)$ under the norm determined by the following $C(\mathbb{T}^n)$-valued inner product:

$$\langle \xi, \eta \rangle_{C(\mathbb{T}^n)}(t) := \sum_{p\in\mathbb{Z}^n} (\overline{\xi}\eta)(t-p). \tag{1}$$

for $t \in \mathbb{R}^n$. We recall further that $\xi \in \Xi$ if and only if ξ is a bounded continuous function on $\mathbb{R}^n$ and $\sum_{p\in\mathbb{Z}^n} |\xi(t-p)|^2$ defines a continuous function on $\mathbb{T}^n$. The right module action of $C(\mathbb{T}^n)$ on Ξ is given by pointwise multiplication, and one can verify that

$$\|\langle \xi, \xi \rangle_{C(\mathbb{T}^n)}\| \;\geq\; \int_{\mathbb{R}^n} |\xi(t)|^2 \, dt, \tag{2}$$

so that $\Xi \;\subseteq\; L^2(\mathbb{R}^n)$. The matrices studied in [20] and [17] were those that were either diagonal, or similar to a diagonal matrix by an element $S \in GL(n,\mathbb{Z})$.

Our approach used the definition of projective multi-resolution analysis for a general $n \times n$ dilation matrix given by M. Rieffel and the author in [20]:

Definition 1. *Fix $n \in \mathbb{N}$, let A be a $n\times n$ integer dilation matrix, and let Ξ be the right-rigged Hilbert $C(\mathbb{T}^n)$ module defined above. A sequence $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ of subspaces of Ξ is called a projective multi-resolution analysis for dilation by A if:*

(i) $\mathcal{V}_0$ *is a finitely generated projective $C(\mathbb{T}^n)$ submodule of Ξ;*
(ii) $\mathcal{V}_j \;=\; D^j(\mathcal{V}_0),\ \forall\, j \;\in\; \mathbb{N}$, *where D is as defined below,*
(iii) $\mathcal{V}_j \;\subset\; \mathcal{V}_{j+1},\ \forall\, j \;\in\; \mathbb{Z}$,
(iv) $\cup_{j=0}^{\infty}\mathcal{V}_j$ *is dense in Ξ, in the Hilbert $C(\mathbb{T}^n)$-module topology,*
(v) $\cap_{-\infty}^{\infty}\mathcal{V}_j \;=\; \{0\}$.

Here D is defined to be the Fourier transformed version of D_A,

$$D \;=\; \mathcal{F}\circ D_A \circ \mathcal{F}^*,$$

where the dilation operator D_A is defined on $L^2(\mathbb{R}^n)$ by

$$D_A(\zeta)(x) \;=\; |\det(A)^{1/2}|\,\zeta(A(x)),\ \zeta \in L^2(\mathbb{R}^n), \tag{3}$$

and the Fourier transform $\mathcal{F}$ is defined by

$$\mathcal{F}(f)(x) \;=\; \int_{\mathbb{R}^n} f(t)\overline{e}(t\cdot x)\, dt, \tag{4}$$

where e is the exponential function defined on $\mathbb{R}$ by $e(r) = e^{2\pi i r}$, and $f \in L^1 \cap L^2(\mathbb{R}^2)$. Extend the Fourier transform to all of $L^2(\mathbb{R}^2)$ in the usual fashion. An easy calculation shows that

$$D \;=\; D_{B^{-1}},$$

for $B = A^t$. It was shown in [20] that condition (v) in the definition above is implied by conditions (i) and (ii) ([20], Proposition 13) and Proposition

14 of [20] showed that if $\mathcal{V}$ were a submodule of Ξ satisfying conditions (i), and if upon setting $\mathcal{V}_j = D^j(\mathcal{V})$, the family $\{\mathcal{V}_j\}$ satisfied (ii) and (iii) of Definition 1, and if in addition there existed $\zeta \in \mathcal{V}$ such that $\zeta(0) \neq 0$, then the family $\{\mathcal{V}_j\}$ would satisfy condition (iv) as well. Indeed, this last fact is very much related to the familiar condition on a scaling function $\phi \in L^2(\mathbb{R})$, that $\hat{\phi}(0) = 1$ (so that $\hat{\phi}(0) \neq 0$).

In [17], following earlier work in [20], we were able to construct projective multi-resolution analyses corresponding to dilation matrices in $GL(n, \mathbb{Z})$ that were of the special type described in the first paragraph.

In this paper, we survey these earlier results, and remark how in the case where the 2×2 dilation matrix is diagonal, whether or not the "wavelet module" $\mathcal{W}_0 = \mathcal{V}_1 \ominus \mathcal{V}_0$ is a free $C(\mathbb{T}^2)$-module depends only on the sign of the determinant. We conjecture that in the 2×2 non-diagonal case, if one starts with an "initial module" $\mathcal{V}_0$ that is not free, whether or not $\mathcal{W}_0$ is free depends only on the sign of the determinant as well.

In the final two sections of the paper, we study the two special cases of the 2×2 integer dilation matrices that are not similar to diagonal integer dilation matrices. One of these matrices is the matrix $Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ associated to the quincunx lattice, and the other is a 2×2 matrix P having determinant -2. It is interesting to note that the results in these two seemingly special cases mirror those in [20] and support our conjecture stated above; that is, in the projective multi-resolution analyses we construct, if the corresponding dilation matrix has positive determinant, the wavelet module $\mathcal{W}_0$ is free even if the initial module $\mathcal{V}_0$ is not free, whereas in the case where the dilation matrix has a negative determinant, when the initial module $\mathcal{V}_0$ is not free, the wavelet module $\mathcal{W}_0$ is also not free. There are some technical difficulties that need to be overcome in these cases that can be generalized to a wider setting, and hence we believe these examples are of some interest.

2 Projective multi-resolution analyses: some motivation and a summary of known results

We first review the notion of ordinary multi-resolution analyses as applied to wavelet theory, as developed by S. Mallat and Y. Meyer. Using the Fourier transform on this concept leads one naturally to consider the projective multi-resolution analysis theory. We concentrate on the two-dimensional case here and in the rest of the article.

We consider ordinary multi-resolution analyses on $L^2(\mathbb{R}^2)$ corresponding to dilation matrices 2×2 integer dilation matrices A, so that both of the eigenvalues of A have modulus greater than 1. We remark at this point that thoughout the paper, we shall denote closed subspaces of Hilbert spaces with ordinary font, rather than the calligraphic font that we have reserved to denote C^*-modules. Indeed, it will often be the case that V_j denotes the closure of a subspace $\mathcal{V}_j$ coming from a projective multi-resolution analysis in the Hilbert space norm, or the Hilbert space closure of the range of $\mathcal{V}_j$ under the inverse Fourier transform.

Definition 2. *A singly generated MRA for dilation by A is a nested sequence $\{V_i\}_{i\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^2)$ satisfying the following conditions:*

(i) *There exists $\phi \in V_0$ such that $\{T_k(\phi) : k \in \mathbb{Z}^2\}$ is an orthonormal basis for V_0, where T_k is translation by $k \in \mathbb{Z}^2$.*
(ii) $V_j = (D_A)^j(V_0)$ *for all j.*
(iii) $V_{j-1} \subset V_j$ *for all j.*
(iv) $\overline{\bigcup_{-\infty}^{\infty} V_j} = L^2(\mathbb{R}^2)$, *i.e. $\bigcup_{-\infty}^{\infty} V_j$ is dense in $L^2(\mathbb{R}^2)$.*
(v) $\bigcap_{-\infty}^{\infty} V_j = \{0\}$.

Here T_k is the unitary operator defined by translation on $L^2(\mathbb{R}^2)$:

$$T_k(f)(t) = f(t-k),\ k \in \mathbb{Z}^2,$$

and D_A is the dilation operator defined on $L^2(\mathbb{R}^2)$ in Equation (3). The main aim of multi-resolution analyses is to construct orthonormal wavelets:

Definition 3. *The set $\{\psi_1, \psi_2, \cdots, \psi_r\} \subseteq L^2(\mathbb{R}^2)$ is called an* **orthonormal wavelet family** *for dilation by the 2×2 matrix A if the set of functions*

$$\cup_{j\in\mathbb{Z}} \cup_{k\in\mathbb{Z}^2} \{D_A^j T_k(\psi_i :\ 1\ \leq i\ \leq r\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^2)$.

We also recall the definition of (normalized) low-pass filter function m_0 in this setting, as it is crucial in the construction of the scaling function.

Definition 4. *Let A be a 2×2 dilation matrix with integer entries and suppose $|det(A)| = d$. A low-pass filter function m_0 for dilation by A is a $\mathbb{Z}^2$ periodic function $m_0 :\ \mathbb{R}^2\ :\to\ \mathbb{C}$ which satisfies the conditions*

(i) *m_0 is continuously differentiable at $(0,0)$ and $m_0(0,0) = 1$.*
(ii) $\sum_{j=0}^{d-1} |m_0((s,t) + \mathbf{v_j})|^2 = 1$, *where $\{\mathbf{v_j}\}_{j=0}^{d-1}$ is a collection of coset representatives in $[A^t]^{-1}(\mathbb{Z}^2)$ for the d-element group $[A^t]^{-1}(\mathbb{Z}^2)/\mathbb{Z}^2$.*
(iii) *m_0 satisfies some form of Cohen's criterion for low-pass filters as outlined in Equation (P') of page 212 of [6], for example.*

Given a low-pass filter m_0 defined as above, the scaling function ϕ can be defined as the inverse Fourier transform of the function

$$\Phi(x)\ =\ \prod_{j=1}^{\infty} m_0([A^t]^{-j}(x)). \tag{5}$$

We remark that if one has a multi-resolution analysis for dilation by A in the sense described in Definition 2, it is possible to find a wavelet family

$\{\psi_1, \psi_2, \cdots, \psi_{d-1}\}$ for $d = |\det(A)|$, where $\{\psi_1, \psi_2, \cdots, \psi_{d-1}\} \subset W_0 = V_1 \ominus V_0$. We describe the procedure briefly.

By means of the Fourier transform, one can transfer the MRAs discussed above over to the frequency domain, and one first builds wavelets there. This was a key insight behind M. Rieffel's initial ideas on projective multi-resolution analyses. So, given an ordinary multi-resolution analysis $\{V_j\}_{j=-\infty}^{\infty}$, consider V_0 in the frequency domain via $\mathcal{F}$, i.e. let us denote all the of the subspaces $\mathcal{F}(V_j)$ by V_j also. Let $\Phi = \mathcal{F}(\phi)$, and suppose that Φ is associated to a low-pass filter m_0 as defined above. Using the fact that $d = |\det(A)|$, it is possible to find $d-1$ functions in $L^\infty(\mathbb{T}^2)$, denoted by $m_1, m_2 \ \cdots, m_{d-1}$, where

$$\{D(m_1\Phi),\ D(m_2\Phi),\ \cdots,\ D(m_{d-1}\Phi)\}$$

are the Fourier transforms of the wavelet family . Here $D = \mathcal{F} \circ D_A \circ \mathcal{F}^{-1}$, and the original wavelet family can be recovered from the formulas

$$\{\psi_i \ = \ \mathcal{F}^{-1}(D(m_i\Phi)) :\ 1 \ \leq i \ \leq d-1\}. \tag{6}$$

Since

$$\mathcal{F}^{-1}(D(m_i\Phi)) \ = \ \mathcal{F}^{-1}\mathcal{F} \circ D_A \circ \mathcal{F}^{-1}(m_i\mathcal{F}(\phi)) \ = \ D_A(\mathcal{F}^{-1}(m_i) * \phi),$$

it is clear that our wavelet family is in the **wavelet subspace** $W_0 = V_1 \ominus V_0$.

This is the observation that motivated Rieffel's original ideas in 1997: under appropriate circumstances, a dense subspace $\mathcal{V}$ of the Fourier transform of the "initial space," $\mathcal{F}(V_0)$, can be viewed as a free, singly generated projective $C(\mathbb{T}^2)$-module, where we view $\mathbb{T}^2$ as $\mathbb{R}^2/\mathbb{Z}^2$.

Since $C(\mathbb{T}^2)$ has finitely generated projective modules which are not free, the first way to generalize the above construction is to take as the "initial" $C(\mathbb{T}^2)$-module a non-free finitely generated projective $C(\mathbb{T}^2)$-module, rather than a free one.

Under the Fourier transform, the main operators we consider are the dilation in the frequency domain D (defined above) and translation conjugated by the Fourier transform, which turns into modulation in the frequency domain, $M_k \ = \ \mathcal{F} \circ T_k \circ \mathcal{F}^{-1}, k \in \mathbb{Z}^2$, which by standard calculation satisfies

$$M_k(f)(x) \ = \ f(x)e(-k \cdot x), f \in L^2(\mathbb{R}^2).$$

The action of the operators $\{M_k : k \in \mathbb{Z}^2\}$ generates a (right) action of the C*-algebra $C(\mathbb{T}^2)$ of continuous functions on the 2-torus, on a Hilbert module Ξ, which we define as follows:

Definition 5. *View $C(\mathbb{T}^2)$, considered as continuous functions on $\mathbb{R}^2$ which are periodic modulo the lattice $\mathbb{Z}^2$, act on the right on $C_c(\mathbb{R}^2)$ as follows:*

$$\xi \cdot f(x) \ = \ \xi(x)f(x),\ \xi \in C_c(\mathbb{R}^2),\ f \in C(\mathbb{T}^2).$$

We define a norm on $C_c(\mathbb{R}^2)$ by setting

$$\|\xi\|^2 \;=\; sup_{x\in\mathbb{T}} \sum_{p\in\mathbb{Z}^2} |\xi(x-p)|^2.$$

If we complete $C_c(\mathbb{R}^2)$ in this norm, we obtain the Hilbert $C(\mathbb{T}^2)$-module Ξ.

Inequality (2) shows that $\Xi \subset L^2(\mathbb{R}^2)$.

Recall in the classical MRA case, the initial space V_0 is formed by taking the closed linear span in $L^2(\mathbb{R}^2)$ of translates by elements of $\mathbb{Z}^2$ of a scaling function ϕ. We have just observed that when we move to the frequency domain, the operation of taking integer translates of a function transforms into multiplying the Fourier transform of the scaling function by a complex exponential function. In the non-free case, our initial module $\mathcal{V}$ will be formed by taking pointwise multiples of a single function $\sigma \in \Xi$, by elements of a certain finitely generated and projective $C(\mathbb{T}^2)$-module. If our initial module is free and singly generated, we will just be multiplying the function σ, which plays the role played by the Fourier transform of the scaling function, by elements in $C(\mathbb{T}^2)$, and the resulting family of functions is a singly generated free projective $C(\mathbb{T}^2)$-module over itself.

Before proceeding further, we consider the structure of the $C(\mathbb{T}^2)$-module Ξ defined above.

Proposition 6. *Let Ξ be the $C(\mathbb{T}^2)$-module defined in Definition 5. Then Ξ is equal to the set of bounded continuous functions ξ on $\mathbb{R}^2$ for which there is some constant $K > 0$ such that*

$$\sum_{p\in\mathbb{Z}^2} |\xi(x-p)|^2 \;\leq\; K,\; \forall x \in \mathbb{R}^2.$$

The periodization of ξ that appears on the left-hand side of the above equation is very familiar, both to wavelet theorists, and to those who construct equivalence bimodules. It also allows us to define the $C(\mathbb{T}^2)$-valued inner product on Ξ, as follows:

$$\langle \xi, \eta\rangle_{C(\mathbb{T}^2)}(x) \;=\; \sum_{p\in\mathbb{Z}^2} \overline{\xi(x-p)}\eta(x-p),\; x \in \mathbb{R}^2/\mathbb{Z}^2. \qquad (7)$$

The definition of projective multi-resolution analysis, first due to M. Rieffel and reviewed in Definition 1, specialized to the case $n = 2$, shows how one can take an increasing union of finitely generated $C(\mathbb{T}^2)$-modules, each one formed from the one preceding it, by applying dilation, and construct Ξ as the norm closure of the union.

We remind the reader of some of the redundancy inherent in Definition 1. In particular, it is not necessary to assume the separation condition (v): Assuming only parts (*i*) and (*ii*) of Definition 1 we obtain the following result:

Proposition 7. *([20], Proposition 13) Let $\mathcal{V}$ be any finitely generated projective $C(\mathbb{T}^2)$-submodule of Ξ, let A be a 2×2 integer dilation matrix with associated dilation operator D, and set $\mathcal{V}_j = D^j(\mathcal{V})$ for all j. Then $\bigcap_{-\infty}^{\infty} \mathcal{V}_j = \{0\}$.*

We note also that the subspaces $\mathcal{V}_j$ for $j \geq 0$ will each themselves be $C(\mathbb{T}^2)$-modules, but the spaces $\mathcal{V}_j$ for $j < 0$ will not be $C(\mathbb{T}^2)$-modules, but only subspaces of $\varXi$. For the purposes of constructing frames in $L^2(\mathbb{R}^2)$, however, all of the $\mathcal{V}_j$ are useful, for both positive and negative j.

We move on to the problem of constructing non-trivial initial modules $\mathcal{V}$. We first show that the density condition (iv) of Definition 1 holds under a natural condition which is closely related to the familiar condition on scaling functions that $\hat{\phi}(0) = 1$.

Proposition 8. *([20], Proposition 14) Let $\mathcal{V}$ be a projective $C(\mathbb{T}^2)$-submodule of $\varXi$ which satisfies conditions $(i), (ii)$ and (iii) of Definition 1. If there is at least one $\xi \in \mathcal{V}$ such that $\xi(0) \neq 0$, then $\mathcal{V}$ satisfies condition (iv) of Definition 1.*

The proof of this result is fairly technical; however it is an extremely convenient result to use, because the main hypothesis is fairly easy to verify in most of our examples.

Let $\mathcal{A}_j = C(\mathbb{R}^2/A^j(\mathbb{Z}^2))$ for $j \in \mathbb{Z}$. Given a projective multi-resolution analysis $\{\mathcal{V}_j\}$ for dilation by the diagonal dilation matrix A, note that for $j \geq 0$, and $k \geq j$, each $\mathcal{V}_k$ is a $\mathcal{A}_j$ module. Moreover, we observe that the dilation operator $D = \mathcal{F} D_A \mathcal{F}^{-1}$ carries $\mathcal{V}_j$ onto $\mathcal{V}_{j+1}$. Let $\mathcal{W}_j$ be the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$ viewed as a $\mathcal{A}_j$-module.

Definition 9. *The $\mathcal{A}_0 = C(\mathbb{T}^2)$-module $\mathcal{W}_0$ constructed as above is called the wavelet module for dilation by A associated to the projective multi-resolution analysis $\{\mathcal{V}_j\}$.*

From now on we denote our initial module by $\mathcal{V} \subseteq \varXi$. In straightforward cases, $\mathcal{V}$ will coincide with the set $\hat{\phi} \cdot C(\mathbb{T}^2)$, for a suitably chosen scaling function ϕ, i.e. it will be a free $C(\mathbb{T}^2)$-module with one generator, $\varPhi = \hat{\phi}$. For example, if one has an ordinary multi-resolution analysis in the time domain generated by one scaling function ϕ whose Fourier transform $\varPhi$ is an element of $\varXi$, then taking $\mathcal{V} = \{\varPhi \cdot g : \ g \in C(\mathbb{T}^2)\} \subseteq \varXi$, and we obtain the desired projective multi-resolution analysis.

Given an ordinary multi-resolution analysis in the time domain where the initial space V_0 has more than one scaling function $\{\phi_1, \phi_2, \cdots, \phi_r\}$ all of whose Fourier transforms lie in $\varXi$, using the same procedure as above, we can construct a free $C(\mathbb{T}^2)$-module $\mathcal{V}$ with r generators. This module approach has proved useful in studying various problems in ordinary wavelet theory. However, one of the aims of this paper is to give a survey of methods of constructing projective multi-resolution analyses whose initial modules $\mathcal{V}$ are not free.

For any finitely generated projective $C(\mathbb{T}^2)$-module, and for any diagonal dilation matrix

$$A = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \ d_1, d_2 \in \mathbb{Z}, |d_i| > 1,$$

Rieffel and the author have shown that it is possible to construct an embedding of this module into Ξ so as to construct a projective multi-resolution analysis for dilation by A.

We review the standard construction non-free finitely generated projective $C(\mathbb{T}^2)$-modules, as described by Rieffel in [22] :

Example 10. For $q \in \mathbb{N}$ and $a \in \mathbb{Z}$, let $X(q,a)$ denote the right $C(\mathbb{T}^2)$-module consisting of the space of continuous complex-valued functions h on $\mathbb{T} \times \mathbb{R}$ which satisfy

$$h(s,t-q) \;=\; e(as)h(s,t), \tag{8}$$

with module action given by

$$h \cdot F(s,t) \;=\; h(s,t)F(s,t), \tag{9}$$

for $h \in X(q,a)$ and $F \in C(\mathbb{T}^2)$. Then $X(q,a)$ is a finitely generated, projective $C^(\mathbb{T}^2)$-module, of rank q and twist $-a$.

There is a $C(\mathbb{T}^2)$-valued inner product defined on $X(q,a)$ which is compatible with the right $C(\mathbb{T}^2)$-action defined by

$$\langle h_1,\ h_2\rangle_{C(\mathbb{T}^2)(s,t)} \;=\; \sum_{k=0}^{q-1}\overline{h_1(s,t-k)}h_2(s,t-k), \tag{10}$$

$h_1, h_2 \in X(q,a)$. The module $X(q,a)$ is **closed** in the norm induced by this inner product.

Moreover, Rieffel has proved the following essential fact concerning his modules $\{X(q,a) :\ q \in \mathbb{N},\ a \in \mathbb{Z}\}$:

Proposition 11. *For $q \in \mathbb{N}$ and $a \in \mathbb{Z}$, let $X(q,a)$ denote the right $C(\mathbb{T}^2)$-module defined above. Then $X(q,a)$ is a finitely generated, projective $C(\mathbb{T}^2)$-module. The set $\{X(q,a) :\ q \in \mathbb{N},\ a \in \mathbb{Z}\}$ parametrizes the isomorphism classes of finitely generated projective $C(\mathbb{T}^2)$-modules, in the sense that if X is a finitely generated projective $C(\mathbb{T}^2)$-module, there exist unique values of q and a such that $X \;\cong\; X(q,a)$.*

Proof. For the proof of this result, refer to [22], Theorem 3.9.

The additive structure of the $\{X(q,a)\}$ is as follows: given $q_1, q_2 \in \mathbb{N}$ and $a_1, a_2 \in \mathbb{Z}$,

$$X(q_1,a_1) \oplus X(q_2,a_2) \;\cong\; X(q_1+q_2, a_1+a_2). \tag{11}$$

Using these results, Rieffel was able to explicitly describe the isomorphism between $K_0(C(\mathbb{T}^2))$ and $\mathbb{Z}^2 \;=\; \{(q,a) :\ q, a \in \mathbb{Z}\}$ ([22]). It was also shown in [22] that with respect to the above parametrization, $K_0(C(\mathbb{T}^2))^+$, the positive cone of $K_0(C(\mathbb{T}^2))$, is equal to $\{[X(q,a)] :\ (q,a) \in \mathbb{N}\ \times \mathbb{Z}\} \;\cup\; \{[0,0]\} \;\subseteq\; \mathbb{Z}^2$. Moreover, Rieffel showed that cancellation holds for finitely generated projective $C(\mathbb{T}^2)$-modules, as follows: if $X(q,a)$ and $X(q',a')$ are given, and if $\mathcal{W}$ is a finitely generated projective $C(\mathbb{T}^2)$-module such that $X(q,a) \oplus \mathcal{W} \cong X(q',a')$, then $\mathcal{W}$ and $X(q'-q, a'-a)$ are isomorphic as $C(\mathbb{T}^2)$-modules.

We now review the main construction of [20]: we want to take any one of the $C(\mathbb{T}^2)$-modules $X(q,a)$ described above and construct a $C(\mathbb{T}^2)$-module monomorphism $\mathcal{R} : X(q,a) \rightarrow \Xi$, so as to obtain the initial module of a projective multi-resolution analysis for dilation by a diagonal dilation matrix.

Theorem 12. *([20], Theorem 4) Let* $d_1,\ d_2$ *be integers whose absolute value is greater than one, and let* $A = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$. *Let* D *denote the operator of dilation by* A *in the frequency domain, i.e.* $D = \mathcal{F} \circ D_A \circ \mathcal{F}^{-1}$. *For every positive integer* q *and each* $a \in \mathbb{Z}$ *there is a* $C(\mathbb{T}^2)$*-module monomorphism* $\mathcal{R} : X(q,a) \rightarrow \Xi$ *which satisfies*
(1) $\langle \mathcal{R}(h_1), \mathcal{R}(h_2) \rangle_{C(\mathbb{T}^2)} = \langle h_1, h_2 \rangle_{C(\mathbb{T}^2)}$, *for all* $h_1\ h_2 \in X(q,a)$, *and*
(2) $\mathcal{R}(X(q,a)) \subseteq D(\mathcal{R}(X(q,a)))$.

Proof. We give only a brief sketch of the proof. The key idea, due to M. Rieffel, is to find construct an appropriate function $\sigma \in \Xi$ and then define

$$\mathcal{R}(h) = h(s,t)\sigma(s,t),\ h \in X(q,a). \tag{12}$$

Conditions (1) and (2) impose restrictions on the choice of the function σ; here to simplify the description, we restrict ourselves to the study of $q = 1,\ a = 1$. The more general conditions are similar, up to change of scale.

In this case, one checks that in order that condition (1) be satisfied, it is necessary to have

$$\sum_{p \in \mathbb{Z}^2} |\sigma((s,t)-p)|^2 = 1,$$

i.e.

$$\langle \sigma, \sigma \rangle_{C(\mathbb{T}^2)} \equiv 1.$$

Then, using calculations with the dilation operator D, one checks that if one wants (2) to hold, there must exist $\widetilde{m}_0 \in X(1,(1-d_1d_2))$ such that

$$\sigma(d_1 s, d_2 t) = \widetilde{m}_0(s,t)\sigma(s,t),$$

(again this looks like ordinary wavelet theory, except for the condition $\widetilde{m}_0 \in X(1,(1-d_1d_2))$ and hence, if one wants (1) and (2) to hold simultaneously, the function $\widetilde{m}_0$ must satisfy $\widetilde{m}_0(0,0) = 1$ and

$$\sum_{j \in \{0,1\cdots|d_1|-1\}} \sum_{k \in \{0,1\cdots|d_2|-1\}} |\widetilde{m}_0(s+\frac{j}{d_1}, t+\frac{k}{d_2})|^2 = 1.$$

We do this by finding an ordinary "low-pass" filter m_0, and then multiplying m_0 by a discontinuous jump function of constant modulus one so that the product will be an element $\widetilde{m}_0$ of $X(1,(1-d_1d_2))$. The points of discontinuity of the jump function will come along the line $t = 1/2$ (where m_0 is equal to 0.) The function σ is constructed in a familiar way:

$$\sigma(s,t) := \Pi_{j=1}^{\infty}\widetilde{m}_0\left(\frac{s}{d_1^j}, \frac{t}{d_2^j}\right), \tag{13}$$

and one can choose $\widetilde{m}_0$ so that σ lies inside Ξ. Indeed one can choose $\widetilde{m}_0$ related to the filters of Y. Meyer so that σ will be continuous and have compact support, or, one can choose $\widetilde{m}_0$ similar to Haar filters, and the paragraphs following Remark 2.2 of [17] show that σ will lie in Ξ in this case as well. Proposition 8 then guarantees that setting $\mathcal{V} = \mathcal{R}(X(q,a))$, we obtain the initial module of a projective multi-resolution analysis.

Corollary 13. *([20], Theorem 6) Let A be the diagonal 2×2 integer dilation matrix and $\mathcal{R} : X(q,a) \to \Xi$ be the module map given in the statement of Theorem 12. Let $\mathcal{V} = \mathcal{R}(X(q,a))$ and define*

$$\mathcal{V}_j \ = \ D^j(\mathcal{V}),\ j \in \mathbb{Z}.$$

Then the family $\{\mathcal{V}_j : \ j \in \mathcal{Z}\}$ is a projective multi-resolution analysis for dilation by A with initial module $\mathcal{V}_0 \cong X(q,a)$.

The modules $\{\mathcal{V}_j : \ j \ \geq \ 0\}$ will all be finitely generated projective $C(\mathbb{T}^2)$-modules. In addition, it is possible to write

$$\Xi \ = \ \mathcal{V} \oplus_{i\geq 0} \ \mathcal{W}_i \text{ (topologically)},$$

where, as before,

$$\mathcal{W}_i \ = \ \mathcal{V}_{i+1} \ominus \mathcal{V}_i.$$

We now discuss the structure of

$$\mathcal{W}_0 \ = \ \mathcal{V}_1 \ominus \mathcal{V}_0,$$

as in ordinary wavelet theory. In fact we will compute the isomorphism class of $\mathcal{W}_0$ as a $C(\mathbb{T}^2)$-module using structure results about $C(\mathbb{T}^2)$-modules described above. (The isomorphism class of $\mathcal{W}_i$ can be determined from knowledge about $\mathcal{W}_0$.)

Theorem 14. *([20], Theorem 8) Let the 2×2 diagonal dilation matrix A with diagonal entries d_1 and d_2, and the module map $\mathcal{R} : X(q,a) \to \Xi$ be as in the statement of Theorem 12, and let $\{\mathcal{V}_j : \ j \in \mathbb{Z}\}$ be the projective multi-resolution analysis with initial module $\mathcal{V}_0$ isomorphic to $X(q,a)$ discussed in Corollary 13. Let $\mathcal{W}_0$ denote the wavelet module $\mathcal{V}_1 \ominus \mathcal{V}_0$. Then $\mathcal{W}_0$ is isomorphic to $X((|det(A)|-1)q, (sign(det(A))-1)a)$ as a $C(\mathbb{T}^2)$-module. In particular, if $det(A)$ is positive, then $\mathcal{W}_0$ is a* **free** *module of dimension $(|det(A)|-1)q$.*

For the proof, we refer the reader to [20], where detailed calculations allow one to prove that

$$\mathcal{V}_1 \cong X(|\det(A)|q, \operatorname{sign}(\det(A))a).$$

Cancellation of finitely generated projective $C(\mathbb{T}^2)$-modules is then used to compute the isomorphism class of the wavelet module.

In [17], the above construction is generalized to give results on diagonal dilation matrices acting on $\mathbb{R}^n$ for $n \geq 3$. We can still form the Hilbert $C(\mathbb{T}^n)$-module Ξ by completing $C_c(\mathbb{R}^n)$ in the appropriate Hilbert module norm, and we construct projective multi-resolution analyses in Ξ_n. For $n \geq 5$, one can no longer use cancellation to calculate the isomorphism class of the wavelet module, but if $n = 3$ or $n = 4$, cancellation still holds. For example, let $n = 3$, and $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. In this case, a projective multi-resolution analysis was constructed with both initial module and wavelet module non-free in [17], even though the determinant of A is positive.

It should be noted that I. Raeburn has mentioned to the author that N. Larsen and he have developed an alterntive approach to the theory of projective multi-resolution analyses from the "direct limit of Hilbert C^*-modules" point of view, just as they developed the theory of ordinary multi-resolution analyses from direct limits of Hilbert space ([12]). It will be interesting to consider any new developments that arise from their approach; Larsen and Raeburn have recently posted a preprint on the ArXiv ([13]).

Remark 15. For completeness, we briefly discuss the relationship between projective multi-resolution analyses and the "multiplicity function" m first constructed by L. Baggett, H. Medina, and K. Merrill in [1]. Given an increasing nested sequence of closed subspaces $\{V_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}^n)$, if V_0 is invariant under the unitary representation of $\mathbb{Z}^n$ on $L^2(\mathbb{R}^n)$ induced by translation, and if under the dilation operator D corresponding to a $n \times n$ dilation matrix A one has $V_n = D^n(V_0)$, $\overline{\cup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R}^n)$ and $\cap_{n\in\mathbb{Z}} V_n = \{\mathbf{0}\}$, we say we have a generalized multi-resolution analysis with core subspace V_0. Using the spectral theory of M. Stone and G. Mackey, the unitary representation of $\mathbb{Z}^n$ on V_0 both gives rise to and is completely determined by a multiplicity function $m : \widehat{\mathbb{Z}^n} \cong \mathbb{T}^n \to \{0, 1, 2, \ldots, \infty\}$, which, roughly speaking, counts the number of times each character $\chi \in \mathbb{T}^n$ occurs in the representation of $\mathbb{Z}^n$ on V_0.

Given a projective multi-resolution analysis $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ of subspaces of Ξ with respect to the $n \times n$ dilation matrix A in the sense of Definition 1, one easily checks by use of the inverse Fourier transform that one obtains a generalized multi-resolution analysis in the sense defined above by setting $V_j = \overline{\mathcal{F}^{-1}(\mathcal{V}_j)}$, for all $j \in \mathbb{Z}$. It is of interest to determine what the multiplicity function m will be corresponding to the representation of $\mathbb{Z}^n$ on $V_0 = \overline{\mathcal{F}^{-1}(\mathcal{V}_0)}$. Indeed, by the discussion in [3], 1.7.1, one can find a projection (self-adjoint idempotent) $P \in M(r, C(\mathbb{T}^n))$ such that $\mathcal{V}_0$ is isomorphic as a projective module to the module $\Gamma(E)$ of continuous sections from $\mathbb{T}^n$ into E, where $(E, \pi_1, \mathbb{T}^n)$ is the vector bundle constructed from P as follows:

$$E \;=\; \{(\mathbf{z}, v) : \mathbf{z} \in \mathbb{T}^n,\; v \in P(\mathbf{z})[\mathbb{C}^r]\},$$

and $\pi_1 : E \;\to\; \mathbb{T}^n$ is projection in the first variable. Since $\mathbb{T}^n$ is connected, E will have constant dimension q for some positive integer $q \leq r$. It follows that $\mathcal{V}_0$ can be viewed as a subspace of the Hilbert space $L^2(\mathbb{T}^n, \mathbb{C}^r)$. If we now close up $\Gamma(E)$ under the Hilbert space norm

$$\|l\|^2 \;=\; \int_{\mathbb{T}^n} \|l(\mathbf{z})\|^2 \, d\mathbf{z},$$

we obtain the Hilbert space of all square-integrable Borel cross-sections from $\mathbb{T}^n$ into E, which we denote by $\Gamma_B(E)$, i.e. $V_0 \;\cong\; \Gamma_B(E)$ can be identified with

$$\{\text{square-integrable Borel } f \;:\; \mathbb{T}^n \;\to\; E :\; \pi_2(f(\mathbf{z})) \;\in\; \text{range} P(\mathbf{z}),\; \forall\, \mathbf{z} \in \mathbb{T}^n\}.$$

This construction is independent of the projection operator P chosen up to conjugacy by a unitary operator. It follows from facts about the Fourier transform that translation by $\mathbf{v}$ in $V_0 \subset L^2(\mathbb{R}^n)$ corresponds to multiplication by a complex exponential function on $\Gamma_B(E)$. Since the fibers $(\pi_1)^{-1}(\chi)$ have the same dimension q for every $\chi \in \mathbb{T}^n$, it follows that the multiplicity of χ is equal to q, for every $\chi \in \mathbb{T}^n$. Thus our projective multi-resolution analyses give rise to generalized multi-resolution analyses with multiplicity functions that are everywhere constant. There do exist generalized multi-resolution analyses with non-constant multiplicity functions; however, the discussion above shows that the topological obstructions that give rise to non-trivial vector bundles disappear when we take the closure in the Hilbert space setting, and that the more unusual non-constant multiplicity functions will not arise from the projective multi-resolution analysis setting. This is not all that surprising, since our complex vector bundles over $\mathbb{T}^n$ will be Borel isomorphic to Cartesian product bundles of the form $\mathbb{T}^n \times \mathbb{C}^q$ for some integer $q \geq 1$.

To date, known results about projective multi-resolution analyses from [20] and [17] concern diagonal matrices with integer entries, and their conjugates by elements of $GL(n, \mathbb{Z})$. In the next sections, we discuss two different examples where the dilation matrices are not similar to diagonal matrices with integer entries.

3 A projective multi-resolution analysis for a non-diagonalizable matrix having determinant −2

We now construct a projective multi-resolution analysis for the matrix P, which is not similar via an element of $GL(2, \mathbb{Z})$ to a diagonal matrix with integer entries. Therefore, such a matrix cannot be studied using the methods of Section 4 of [17].

We consider the matrix given by $P = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. This matrix has eigenvalues $\sqrt{2}i$ and $-\sqrt{2}i$, so is a dilation matrix, and cannot be similar by an element of $SL(2,\mathbb{Z})$ to a diagonal matrix. Since the determinant of P is equal to -2, if we tried to generalize the results of Theorem 7 of [20], and construct a projective multi-resolution analysis in $\Xi \subseteq L^2(\mathbb{R}^2)$ with initial module $\mathcal{V}_0$ isomorphic to $X(1,1)$, the conjecture that was stated in the last paragraph of the Introduction would be that the dilated module $D(\mathcal{V}_0) = \mathcal{V}_1$ would be isomorphic to $X(2,-1)$, so that by cancellation, the wavelet module would be isomorphic to $X(1,-2) = X(1, \operatorname{sign}(\det(P)) - 1)$. We shall verify the conjecture in this case by constructing the projective multi-resolution analysis and computing the class in $K_0(C(\mathbb{T}^2))$ of the wavelet module $\mathcal{W}_0$.

Although the methods of [20] and [17] cannot be applied directly, we will use some of the same general principles: first we will construct the filter functions, corresponding to an ordinary multi-resolution analysis constructed for the matrix P, and then we will modify these filter functions so as to construct the desired embedding of a non-free finitely generated projective $C(\mathbb{T}^2)$-module into Ξ. This will give us the initial space $\mathcal{V}_0$ in the desired projective multi-resolution analysis. Some of the techniques employed in constructing the isomorphisms which follow in this section and in Section 4 have been used before in a previous paper by the author dealing with the construction of projective modules for C^*-algebras related to the discrete Heisenberg group [16].

We first construct a low-pass filter for dilation by P as defined in Definition 4. Since

$$[P^t]^{-1} \ = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},$$

we choose for our coset representatives of the two-element group $[P^t]^{-1}(\mathbb{Z}^2)/\mathbb{Z}^2$ the elements in the set $\{(0,0),\ (0,-1/2)\}$.

Thus if we take any of the standard low-pass filter functions of one variable t corresponding to dilation on by 2 and extend this to a function, denoted by m_0, defined on $\mathbb{R}^2$ by letting it be constant in the s variable, one obtains an ordinary scaling function ϕ corresponding to dilation by P defined in the usual way by

$$\hat{\phi}(s,t) \ = \ \Phi(s,t) \ = \ \prod_{j=1}^{\infty} m_0([P^t]^{-j}(s,t)).$$

So, for example, we can take the low-pass filter corresponding to the Haar wavelet

$$m_0(s,t) \ = \ \frac{1}{2}(1+e(t)),$$

or any other low-pass filter in the variable t, e.g. we could take a low-pass filter in the t variable giving rising to a Meyer wavelet, whose scaling function is compactly supported in the frequency domain.

Now as in Equation (12), we want to find a function $\sigma^- \in \Xi$ and then set

$$\mathcal{R}^-(h) = h \cdot \sigma^-, \ h \in X(1,1), \tag{14}$$

and we want to choose σ^- in such a way that
(1) $\langle \mathcal{R}^-(h_1), \mathcal{R}^-(h_2)\rangle_{C(\mathbb{T}^2)} = \langle h_1, h_2\rangle_{C(\mathbb{T}^2)}$, for all $h_1, \ h_2 \in X(1,1)$,
and
(2) $\mathcal{R}^-(X(1,1)) \subseteq D[\mathcal{R}^-(X(1,1))]$,
where now D denotes the Fourier transformed version of D_P, for $P = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. We denote these maps with "−" superscripts, as the matrix P under consideration has negative determinant -2, and in the next section we shall carry out a similar exercise for a matrix Q corresponding to the quincunx lattice that has positive determinant 2; the maps in that section will be decorated with "+" superscripts.

One checks that, as in Theorem 4 of [20], in order to have (1) it is necessary and sufficient to have

$$\langle \sigma^-, \sigma^- \rangle_{C(\mathbb{T}^2)} = 1,$$

where, as in Equation (13), we will set

$$\sigma^-(s,t) = \prod_{j=1}^{\infty} \widetilde{m}_0([P^t]^{-j}(s,t)), \tag{15}$$

for an appropriately chosen function $\widetilde{m}_0$.

Recall from Section 4 of [20] the jump function $J_{q,a,\beta} : \mathbb{R}^2 \to \mathbb{C}$ defined by

$$J_{q,a,\beta}(s,t) = \ e(jas) \text{ for } \beta + jq \le t < \beta + (j+1)q, \ q, a, j \ \in \mathbb{Z}, \ \beta > 0. \tag{16}$$

We have $J_{q,a,\beta}(0,0) = 1$ and $J_{q,a,\beta}(s, t-q) = e(-as)J_{q,a,\beta}(s,t)$. (This formula is slightly different from that given in [20], Section 4, but a slight change in subscripts in formulas involving the jump functions in [20] is all that is needed to modify the proofs there.) Note that $J_{q,a,\beta}$ has jump discontinuities in the t-variable along the lines $t = \beta + jq, \ j \in \mathbb{Z}$, but is continuous elsewhere. Hence, if one multiplies $J_{q,a,\beta}$ by a function in the variables s and t that vanishes along all of those lines, the resulting product function will be continuous in s and t simultaneously.

Calculations lead us to choose

$$\widetilde{m}_0(s,t) = e(2st)J_{1,-3,-1/2}(s,t)m_0(t), \tag{17}$$

where the jump function $J_{1,-3,-1/2}(s,t)$ is defined as stated above:

$$J_{1,-3,-1/2}(s,t) = e(-3js)$$

for $j-1/2 \leq t < j+1/2,\ j \in \mathbb{Z}$. As before one checks that $J_{1,-3,-1/2}(0,0) = 1$ and $J_{1,-3,-1/2}(s,t-1) = e(3s)J_{1,-3,-1/2}(s,t)$, so that $\sigma^-(s,t) = \prod_{j=1}^{\infty} \widetilde{m}_0([P^t]^{-j}(s,t))$, and

$$\sigma^-(2t,s) \ = \ \sigma^-([P^t]^{-1}(s,t)) \ = \ \widetilde{m}_0(s,t)\sigma^-(s,t).$$

Then, using the fact that $D = D_{[P^t]^{-1}}$, we see that in order for (2) to hold, it is necessary that

$$D_{P^t}(\mathcal{R}^-(X(1,1)) \ \subseteq \ \mathcal{R}^-(X(1,1)), \tag{18}$$

that is,

$$\{\sqrt{2}\sigma^-(2t,s)h(2t,s) : \ h \ \in \ X(1,1)\} \ \subseteq \ \{\sigma^-(s,t)f(s,t) : \ f \ \in \ X(1,1)\}.$$

Thus, given $h \in X(1,1)$, we want to find $g \in X(1,1)$ such that

$$\sqrt{2}\sigma^-(2t,s)h(2t,s) \ = \ \sigma^-(s,t)g(s,t).$$

But

$$\begin{aligned}\sqrt{2}\sigma^-(2t,s)h(2t,s) \ &= \ \sqrt{2}\widetilde{m}_0(s,t)\sigma^-(s,t)h(2t,s) \\ &= \ \sigma^-(s,t)[\sqrt{2}\widetilde{m}_0(s,t)h(2t,s)].\end{aligned}$$

Thus we take the obvious choice

$$g(s,t) \ = \ \sqrt{2}\widetilde{m}_0(s,t)h(2t,s),$$

and we now show that g as defined above is in $X(1,1)$. We first note

$$g(s-1,t) \ = \sqrt{2}\widetilde{m}_0(s-1,t)h(2t,s-1)$$

$$= \ e(-2t)e(2t)\sqrt{2}e(2st)J_{1,-3,1/2}(s,t)m_0(t)e(2t)h(2t,s) = \ g(s,t),$$

so that g is $\mathbb{Z}$-periodic in the s-variable. Now we check what happens under $\mathbb{Z}$-translation in the t-variable.

$$g(s,t-1) \ = \ \sqrt{2}\widetilde{m}_0(s,t-1)h(2(t-1),s)$$

$$= \ \sqrt{2}e(-2s)e(3s)J_{1,3,1/2}(s,t)m_0(t)h(2t,s) \ = \ e(s)g(s,t).$$

Therefore, $g \in X(1,1)$, and we have shown that the inclusion (18) holds, so that Condition (2) is verified.

The above results have proved most of the following:

Theorem 16. *There is a $C(\mathbb{T}^2)$-module monomorphism $\mathcal{R}^- : \ X(1,1) \ \to \ \Xi$ which satisfies*
(1) $\langle \mathcal{R}^-(h_1), \mathcal{R}^-(h_2)\rangle_{C(\mathbb{T}^2)} \ = \ \langle h_1, h_2\rangle_{C(\mathbb{T}^2)}$, *for all* $h_1,\ h_2 \ \in \ X(1,1)$,
and
(2) $\mathcal{R}^-(X(1,1)) \ \subseteq \ D[\mathcal{R}^-(X(1,1))]$,

where D denotes the Fourier transformed version of D_P, and P is the matrix of determinant -2 discussed above, i.e. $P = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. *Furthermore, setting* $\mathcal{V}_j = D^j(\mathcal{R}^-(X(1,1))$ *for* $j \geq 0$, *the resulting family of subspaces* $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ *of* Ξ *is a projective multi-resolution analysis for dilation by* Q *with initial module* $\mathcal{V}_0$ *isomorphic to* $X(1,1)$.

Proof. All we have left to show is that $\{\mathcal{V}_j = D^j(\mathcal{R}^-(X(1,1)))\}_{j=-\infty}^{\infty}$ is a projective multi-resolution analysis in the sense of [20]. But this follows from Proposition 8, since obviously $\mathcal{R}^-(X(1,1))$ contains an element ξ with $\xi(0,0) \neq 0$.

We now consider $\mathcal{V}_1 = D((\mathcal{R}^-(X(1,1))) = D_{[P^t]^{-1}}((\mathcal{R}^-(X(1,1)))$

$$= \{\frac{1}{\sqrt{2}}\sigma^-(t,s/2)h(t,s/2) : h \in X(1,1)\}.$$

We will show there is an isomorphism between $\mathcal{V}_1$ and $X(2,-1)$, thus giving an example where Theorem 14 generalizes to a non-diagonal matrix, supporting our conjecture as stated at the end of the introduction of this article.

We define the map $\rho^- : \mathcal{V}_1 = D((\mathcal{R}^-(X(1,1))) \to X(2,-1)$ as follows:

$$\rho^-(\frac{1}{\sqrt{2}}(\sigma^-\cdot h)\circ[P^t]^{-1})(s,t) = r(s,t)h(t,s/2)+r(s-1,t)h(t,(s-1)/2), \tag{19}$$

where $r : \mathbb{R}^2 \to \mathbb{C}$ is a continuous function to be determined.

In order that $\rho^-(\frac{1}{\sqrt{2}}(\sigma^- \cdot h) \circ [P^t]^{-1})(s,t) \in X(2,-1)$, we need to have $\mathbb{Z}$-periodicity in the s variable, i.e. we need

$$\rho^-(\frac{1}{\sqrt{2}}(\sigma^- \cdot h) \circ [P^t]^{-1})(s-1,t) = \rho^-(\frac{1}{\sqrt{2}}(\sigma^- \cdot h) \circ [P^t]^{-1})(s,t), \tag{20}$$

and we need a functional equation satisfied in the t-variable:

$$\rho^-(\frac{1}{\sqrt{2}}(\sigma^- \cdot h) \circ [P^t]^{-1})(s,t-2) = e(-s)\rho^-(\frac{1}{\sqrt{2}}(\sigma^- \cdot h) \circ [P^t]^{-1})(s,t), \tag{21}$$

One checks that Equation (20) will hold only when

$$r(s-2,t) = e(-t)r(s,t)\forall s,t \in \mathbb{R}, \tag{22}$$

and Equation (21) will hold if and only if

$$r(s,t-2) = e(-s)r(s,t) \ \forall s,t \in \mathbb{R}. \tag{23}$$

Thus, we want to find continuous $r : \mathbb{R}^2 \to \mathbb{C}$ that satisfy the two conditions given in Equations (22) and (23).

A simple calculation shows that if we take

$$r(s,t) \;=\; \frac{1}{2}e(st/2), \tag{24}$$

Equations (22) and (23) will be satisfied. The constant multiple $\frac{1}{2}$ has been introduced to make future inner product calculations work out correctly.

Thus we define

$$\rho^-((\frac{1}{\sqrt{2}}(\sigma^-\cdot h)\circ[P^t]^{-1})(s,t) \;=\; \frac{e(st/2)}{2}[h(t,\frac{s}{2})+e(-t/2)h(t,\frac{s-1}{2})], \tag{25}$$

for all $h\in X(1,1)$, and we have shown that the image of ρ^- is contained in $X(2,-1)$, as desired. We now are ready to complete the proof of the following:

Theorem 17. *Let* $\mathcal{V}_0 \;=\; \mathcal{R}^-(X(1,1))$ *be the projective* $C(\mathbb{T}^2)$ *submodule of* Ξ *isomorphic to* $X(1,1)$ *constructed in Theorem 16, so that* $\mathcal{V}_1 \;=\; D(\mathcal{V}_0) \;\supseteq\; \mathcal{V}_0$, *for the dilation corresponding to* $P \;=\; \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. *Then* $\mathcal{V}_1 \;\cong\; X(2,-1)$, *so that*

$$\mathcal{W}_0 \;=\; \mathcal{V}_1\ominus\mathcal{V}_0 \;\cong\; X(1,-2).$$

i.e. $\mathcal{W}_0$ *is a finitely generated projective* $C(\mathbb{T}^2)$*-module of dimension* 1 *and twist* 2.

Proof. We have constructed a map ρ^- of $\mathcal{V}_1 = \;D(\mathcal{R}^-(X(1,1)))$ into $X(2,-1)$ above, and clearly ρ^- is a right $C(\mathbb{T}^2)$-module homomorphism. We need to show that this map preserves the $C(\mathbb{T}^2)$-valued inner products and is bijective.

We consider the inner product in $\mathcal{V}_1$ viewed as a submodule of Ξ first:

$$\langle D(\mathcal{R}^-(h_1)), D(\mathcal{R}^-(h_2))\rangle_{C(\mathbb{T}^2)}(s,t)$$

$$= \;\frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2}\overline{(\sigma^-\cdot h_1)\circ[P^t]^{-1}((s,t)-(m,n))}(\sigma^-\cdot h_2)\circ[P^t]^{-1}((s,t)-(m,n))$$

$$= \;\frac{1}{2}\sum_{(m',n')\in[P^t]^{-1}(\mathbb{Z}^2)}|\sigma^-((t,\frac{s}{2})-(m',n'))|^2\overline{h_1}\cdot h_2((t,\frac{s}{2})-(m',n')).$$

We break this sum up into two terms, using the two coset representatives $(0,0)$ and $(0,-\frac{1}{2})$ for the two-element quotient group $(P^t)^{-1}(\mathbb{Z}^2)/(\mathbb{Z}^2)$, obtaining as our inner product

$$\frac{1}{2}\langle\sigma^-\cdot h_1,\sigma^-\cdot h_2\rangle_{C(\mathbb{T}^2)}(t,\frac{s}{2}) \;+\; \frac{1}{2}\langle\sigma^-\cdot h_1,\sigma^-\cdot h_2\rangle_{C(\mathbb{T}^2)}((t,\frac{s}{2})+(0,-\frac{1}{2})),$$

which is equal to

$$\frac{1}{2}[\overline{h_1(t,\frac{s}{2})}h_2(t,\frac{s}{2})+\overline{h_1(t,\frac{s-1}{2})}h_2(t,\frac{s-1}{2})]. \tag{26}$$

We now consider the inner product of $\rho^-(D(\mathcal{R}^-(h_1)))$ and $\rho^-(D(\mathcal{R}^-(h_2)))$ in $X(2,-1)$. Recall if $f_1,\ f_2\in X(2,-1)$, the inner product is given by

$$\langle f_1, f_2\rangle_{C(\mathbb{T}^2)}(s,t) \;=\; \sum_{j=0}^{1}\overline{f_1(s,t-j)}f_2(s,t-j).$$

Thus

$$\langle \rho^-(\frac{1}{\sqrt{2}}(\sigma\cdot h_1)), \rho^-(\frac{1}{\sqrt{2}}(\sigma\cdot h_1))\rangle_{C(\mathbb{T}^2)}(t,s/2)$$

$$= \frac{1}{4}\sum_{j=0}^{1}\overline{(e(s(t-j)/2)h_1(t-j,s/2)+e((s-1)(t-j)/2)h_1(t-j,(s-1)/2))}\times\cdot$$

$$\times(e(s(t-j)/2)h_2(t-j,s/2)+e((s-1)(t-j)/2)h_2(t-j,(s-1)/2))$$

$$= \frac{1}{4}[(\overline{h_1}\cdot h_2)(t,s/2)+(\overline{h_1}\cdot h_2)(t,(s-1)/2)+e(-t/2)\overline{h_1(t,s/2)}h_2(t,(s-1)/2)]$$

$$+\frac{1}{4}[e(t/2)(\overline{h_1}\cdot h_2)(t,(s-1)/2)+(\overline{h_1}\cdot h_2)(t-1,s/2)]$$

$$+\frac{1}{4}[(\overline{h_1}\cdot h_2)(t,(s-1)/2)+e(-(t-1)/2)\overline{h_1(t-1,s/2)}h_2(t-1,(s-1)/2)]$$

$$+\frac{1}{4}e((t-1)/2)(\overline{h_1}\cdot h_2)(t-1,(s-1)/2),$$

which, using the fact that h is $\mathbb{Z}$-periodic in the first variable, and the fact that $e(-1/2)=e(1/2)=-1$, gives us

$$\frac{1}{2}[(\overline{h_1}\cdot h_2)(t,s/2)+(\overline{h_1}\cdot h_2)(t,(s-1)/2)]. \tag{27}$$

By comparing the quantities in Expressions (26) and (27), we see we have shown that

$$\langle D(\mathcal{R}^-(h_1)), D(\mathcal{R}^-(h_2))\rangle_{C(\mathbb{T}^2)} \;=\; \langle \rho^-(D(\mathcal{R}^-(h_1))), \rho^-(D(\mathcal{R}^-(h_2)))\rangle_{C(\mathbb{T}^2)}, \tag{28}$$

so that ρ^- preserves the $C(\mathbb{T}^2)$ valued inner product.

Finally, we need to show that ρ^- is surjective. Define a map $\varUpsilon : X(2,-1)\to C_b(\mathbb{T},\mathbb{R})$ by

$$\varUpsilon(g)(s,t) \;=\; e(-(st))g(2t,s)+e(-(s-1)t)g(2t,s-1).$$

Typical calculations show that in fact $\varUpsilon : X(2,-1)\to X(1,1)$, and that

$$\rho^-((\frac{1}{\sqrt{2}}\sigma^-\circ[P^t]^{-1})\cdot(\varUpsilon(g)\circ[P^t]^{-1}))(s,t) \;=\; g(s,t), \tag{29}$$

and since g was an arbitrary element of $X(2,-1)$ we have shown that ρ^- is surjective.

The fact that $\mathcal{W}_0$ is isomorphic to $X(1,-2)$ follows directly from our identification of $\mathcal{V}_1$ and the cancellation property for finitely generated projective $C(\mathbb{T}^2)$-modules.

4 A projective multi-resolution analysis for the matrix associated to the quincunx lattice

The matrix discussed in the previous section, $P = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ is of interest in that it has integer determinant -2, yet is not similar via a matrix with integer entries to a diagonal matrix. We now consider another matrix of this type, $Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. The matrix Q is also not similar via an element of $GL(2,\mathbb{Z})$ to a diagonal matrix with integer entries, so it too cannot be studied using the methods of Section 4 of [17]. We will construct a projective multi-resolution analysis corresponding to dilation by the unitary operator constructed from Q by means similar to those used in Section 3.

Recall the quincunx lattice refers to the lattice in $\mathbb{R}^2$ formed from the columns of the 2×2 dilation matrix

$$Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

i.e. the lattice in $\mathbb{R}^2$ with basis vectors $\mathbf{q_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{q_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The eigenvalues of Q are $1 \pm i$, so that Q cannot be similar to a matrix with integral diagonal entries via an element of $GL(2,\mathbb{C})$, thus certainly not through an element of $SL(2,\mathbb{Z})$. We now construct a projective multi-resolution analysis in $L^2(\mathbb{R}^2)$ corresponding to this dilation matrix with $\mathcal{V}_0 \cong X(1,1)$, and calculate that the corresponding wavelet module $\mathcal{W}_0$ is isomorphic to $X(1,0) \cong C(\mathbb{T}^2)$. Although the methods of [20] and [17] cannot be applied directly, we will use some of the same general principles that were used in the previous section. First we will construct the filter functions, or masks, corresponding to an ordinary multi-resolution analysis coming from the quincunx dilation matrix, and then we will modify these filter functions so as to construct the desired embedding of a non-free finitely generated projective $C(\mathbb{T}^2)$-module into Ξ. This will give us the initial space $\mathcal{V}_0$ in the projective multi-resolution analysis that we want.

We first construct a low-pass filter for dilation by Q as defined in Definition 4. Since

$$[Q^t]^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

we choose for our coset representatives of $[Q^t]^{-1}(\mathbb{Z}^2)/\mathbb{Z}^2$ the two element set $\{(0,0),\ (1/2,-1/2)\}$. Then the equation in *(ii)* of Definition 4 becomes

$$|m_0(s,t)|^2 + |m_0(s+1/2,t-1/2)|^2 = 1,$$

and Cohen has verified on p. 213 of [6] that if we take any of the standard low-pass filter functions of one variable s corresponding to dilation on by 2 and extend this to a function, denoted by m_0, defined on $\mathbb{R}^2$ by letting it be constant in the t variable, one obtains an ordinary scaling function ϕ corresponding to dilation by Q defined in the usual way by

$$\hat{\phi}(s,t) \;=\; \Phi(s,t) \;=\; \prod_{j=1}^{\infty} m_0([Q^t]^{-j}(s,t)).$$

So, as in Section 3, we can take the low-pass filter corresponding to the Haar wavelet

$$m_0(s,t) \;=\; \frac{1}{2}(1+e(s)),$$

or any other low-pass filter in the variable s whose corresponding infinite product lies in Ξ. The most important fact in what follows is that the function m_0 vanishes all along the line $s \;=\; \frac{1}{2}$.

As in the previous section, we want to find a function $\sigma^+ : \;\mathbb{R}^2 \;\to\; \mathbb{C}$ such that the map

$$\mathcal{R}^+ \;: X(1,1) \;\to\; \Xi$$

defined by

$$\mathcal{R}^+(h) \;=\; \sigma^+ \cdot h,\; h \;\in\; X(1,1),$$

preserves the $C(\mathbb{T}^2)$- module structure and the $C(\mathbb{T}^2)$-valued inner product. As mentioned in the previous section, we denote these maps with "+" superscripts, as the matrix corresponding to the quincunx lattice has positive determinant +2. Keeping close to our notation of the previous section, we will denote the image of $\mathcal{R}^+$, $\mathcal{R}^+(X(1,1))$, by $\mathcal{V}_0$, and since we want to have

$$\mathcal{V}_0 \;\subseteq\; D(\mathcal{V}_0),$$

the usual calculations involving Fourier transforms lead us to choose σ^+ so that

$$D_{Q^t}(\mathcal{V}_0) \;\subseteq\; \mathcal{V}_0.$$

Here, as in the discussion following Definition 1, D denotes the Fourier transformed version of D_Q.

We look for conditions on the function σ^+ that will guarantee that

$$\sigma^+(s-t,s+t)f(s-t,s+t) \;\;\in\; \sigma^+ \cdot X(1,1),\; \forall\, f \;\in\; X(1,1).$$

As in [20], we choose σ^+ so that it satisfies a dilation equation, this time the equation

$$\sigma^+(s-t,s+t) \;=\; \widetilde{m}_0(s,t)\sigma^+(s,t),$$

where $\widetilde{m}_0$ is a modified form of the low-pass filter m_0 multiplied by jump functions of various sorts which have constant modulus one on $\mathbb{R}$. We state the theorem:

Theorem 18. *There is a* $C(\mathbb{T}^2)$*-module monomorphism* $\mathcal{R}^+ :\ X(1,1) \to \Xi$ *that satisfies*
$(1')$ $\langle \mathcal{R}^+(h_1), \mathcal{R}^+(h_2)\rangle_{C(\mathbb{T}^2)} \ =\ \langle h_1, h_2\rangle_{C(\mathbb{T}^2)}$, *for all* $h_1\ h_2\ \in\ X(1,1)$,
and
$(2')$ $\mathcal{R}^+(X(1,1))\ \subseteq\ D[\mathcal{R}^+(X(1,1))]$,
where D *denotes the Fourier transformed version of* D_Q, *and* Q *is the matrix associated to the quincunx lattice, i.e.* $Q\ =\ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. *Furthermore, setting* $\mathcal{V}_i\ =\ D^i(\mathcal{R}^+(X(1,1))$ *for* $i \in \mathbb{Z}$, *the resulting family of subspaces* $\{\mathcal{V}_i\}_{i\in\mathbb{Z}}$ *of* Ξ *is a projective multi-resolution analysis for dilation by* Q *with initial module* $\mathcal{V}_0$ *isomorphic to* $X(1,1)$.

Proof. As mentioned above, we need to find an appropriate function $\sigma^+\ \in\ \Xi$ and then set

$$\mathcal{R}^+(h)\ =\ h\cdot\sigma^+,\ h\ \in\ X(1,1).$$

As in the previous section, in order to have $(1')$ it is necessary and sufficient to have

$$\langle \sigma^+, \sigma^+\rangle_{C(\mathbb{T}^2)} = 1. \tag{30}$$

Then, using the fact that $D = D_{[Q^t]^{-1}}$, we see that in order for $(2')$ to hold, it is necessary that

$$D_{Q^t}(\mathcal{R}^+(X(1,1))\ \subseteq\ \mathcal{R}^+(X(1,1)), \tag{31}$$

so that

$$\{\sqrt{2}\sigma^+(s-t,s+t)h(s-t,s+t) :\ h\ \in\ X(1,1)\}\ \subseteq\ \{\sigma^+(s,t)f(s,t) :\ f\ \in\ X(1,1)\}.$$

If $h\ \in\ X(1,1)$, then h will satisfy the two equalities

$$h(s-1-t, s-1+t)\ =\ e(s-t)h(s-t,s+t),$$

and

$$h(s-(t-1), s+(t-1))\ =\ e(s-t)h(s-t,s+t).$$

A routine calculation shows that a sufficient condition for $g(s,t) := \sqrt{2}\sigma^+(s-t, s+t)h(s-t,s+t)$ to be an element of $\mathcal{R}^+(X(1,1))$ for every $h\ \in\ X(1,1)$ is that there should exist $\widetilde{m}_0 :\ \mathbb{R}^2\ \to\ \mathbb{C}$ such that

$$\widetilde{m}_0(s,t-1)\ =\ e(t)\widetilde{m}_0(s,t) \tag{32}$$

and

$$\widetilde{m}_0(s-1,t)\ =\ e(t-s)\widetilde{m}_0(s,t) \tag{33}$$

satisfying

$$\sigma^+(s-t,s+t)\ =\ \widetilde{m}_0(s,t)\sigma^+(s,t).$$

Calculations similar to those done in [20] show that in order for $(1')$ to hold, it is necessary and sufficient that $\widetilde{m}_0(0,0)\ =\ 1$,

$$|\widetilde{m}_0(s,t)|^2 \;+\; |\widetilde{m}_0(s+1/2,t+1/2)|^2 \;=\; 1,$$

and the filter $\widetilde{m}_0$ satisfy some form of Cohen's non-vanishing condition around the origin in $\mathbb{R}^2$. Keep in mind that our proposed candidate for $\widetilde{m}_0$ will be a function of modulus one times any of the low-pass filter functions m_0 for dilation by 2, which were shown by Cohen to satisfy his non-vanishing condition, so that $\widetilde{m}_0$ will certainly satisfy the non-vanishing condition also.

Recall from Equation 16 that the jump function $J_{1,-1,-1/2}(s,t)$ is given by by $J_{1,-1,-1/2}(s,t) \;=\; e(-js)$ for $j-1/2 \;\leq\; t \;<\; j+1/2,\; j \;\in\; \mathbb{Z}$. We know that $J_{1,-1,-1/2}(0,0)=1$ and $J_{1,-1,-1/2}(s,t-1) \;=\; e(s)J_{1,-1,-1/2}(s,t)$. Define

$$\widetilde{J}(s,t) \;=\; J_{1,-1,-1/2}(t-s,s).$$

Then standard calculations show that $\widetilde{J}(s-1,t) \;=\; e(t-s)\widetilde{J}(s,t)$ and $\widetilde{J}(s,t-1) \;=\; \widetilde{J}(s,t)$. The function $\tilde{J}$ has discontinuities along the line $s \;=\; \pm\frac{1}{2}$. We now use a technique similar to one used in Lemma 1.3 of [16] and define

$$q(t) \;=\; e(\frac{-[t^2+t]}{2}). \tag{34}$$

Observe that $q(0)=1$ and $q(t-1) \;=\; e(t)q(t)$. Finally, set

$$\widetilde{m}_0(s,t) \;=\; q(t)\widetilde{J}(s,t)m_0(s), \tag{35}$$

where m_0 is any continuously differentiable low-pass filter function for dilation by 2, such that the Fourier transform of the corresponding scaling function lies in $\varXi$. We note that $\widetilde{m}_0$ is continuous, because along the line of discontinuity $s \;=\; \pm\frac{1}{2}$ of $\tilde{J}$ we have $m_0 \;=\; 0$. Also, by construction,

$$\widetilde{m}_0(0,0) \;=\; 1,$$

$$\widetilde{m}_0(s,t-1) \;=\; e(t)\widetilde{m}_0(s,t),$$
$$\widetilde{m}_0(s-1,t) \;=\; e(t-s)\widetilde{m}_0(s,t),$$

and since $\widetilde{m}_0$ has the same modulus as m_0, $\widetilde{m}_0$ satisfies Cohen's non-vanishing condition and

$$|\widetilde{m}_0(s,t)|^2 \;+\; |\widetilde{m}_0(s+\frac{1}{2},t+\frac{1}{2})|^2 \;=\; 1.$$

Again, as in Equation (13) we set

$$\sigma^+(s,t) \;=\; \prod_{j=1}^{\infty}\widetilde{m}_0([Q^t]^{-j}(s,t)); \tag{36}$$

it follows that $\sigma^+(s,t)$ is continuous and

$$\sigma^+(Q^t(s,t)) \;=\; \widetilde{m}_0(s,t)\sigma^+(s,t).$$

Furthermore, we know that

$$\langle \sigma^+, \sigma^+ \rangle_{C(\mathbb{T}^2)}(s,t) = 1$$

everywhere on $\mathbb{R}^2$, and not just almost everywhere, by our choice of m_0 that has the same modulus as $\widetilde{m}_0$. Defining

$$\mathcal{R}^+(h)(s,t) \; = \; \sigma^+(s,t)h(s,t),$$

we thus obtain the desired $C(\mathbb{T}^2)$-module monomorphism of $X(1,1)$ into Ξ which preserves the $C(\mathbb{T}^2)$-valued inner products, and which, in addition, satisfies the condition

$$\mathcal{R}^+(X(1,1)) \; \subseteq \; D[\mathcal{R}^+(X(1,1))].$$

Setting

$$\mathcal{V}_j \; = \; D^j[\mathcal{R}^+(X(1,1))]$$

for $j \in \mathbb{Z}$, an application of Theorem 6 of [20] shows that the family $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ is the desired projective multi-resolution analysis.

Our aim now is to compute the isomorphism type of the wavelet module

$$\mathcal{W}_0 \; = \; \mathcal{V}_1 \; \ominus \; \mathcal{V}_0$$

which arises from Theorem 18. We note that the determinant of Q is 2, which is positive. Our conjecture as stated in the last paragraph of the introductory section leads us to believe that $\mathcal{V}_1 \;\cong\; X(2,1)$, and hence that $\mathcal{W}_0 \;\cong\; X(1,0)$, a free module of rank 1 over $C(\mathbb{T}^2)$. This turns out to be true, although the proof is somewhat technical in nature:

Theorem 19. *Let* $\mathcal{V}_0 \; = \; \mathcal{R}^+(X(1,1))$ *be the projective* $C(\mathbb{T}^2)$*-submodule of* Ξ *isomorphic to* $X(1,1)$ *constructed in Theorem 18, so that* $\mathcal{V}_1 \; = \; D(\mathcal{V}_0) \; \supseteq \; \mathcal{V}_0$*, for* $Q \; = \; \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. *Then* $\mathcal{V}_1 \;\cong\; X(2,1)$*, so that*

$$\mathcal{W}_0 \; = \; \mathcal{V}_1 \ominus \mathcal{V}_0 \;\cong\; X(1,0),$$

i.e. $\mathcal{W}_0$ *is a finitely generated free* $C(\mathbb{T}^2)$*-module of dimension* 1.

Proof. Recall that

$$\mathcal{V}_1 \; = \; \{\frac{1}{\sqrt{2}}(\sigma^+ \cdot h)(\frac{s+t}{2}, \frac{-s+t}{2}) : \; h \; \in \; X(q,a)\}, \tag{37}$$

where

$$\sigma^+(s-t, s+t) \; = \; \widetilde{m_0}(s,t)\sigma^+(s,t),$$

for $\widetilde{m_0} \; : \; \mathbb{R}^2 \; \to \; \mathbb{C}$ constructed as in Theorem 18, so that σ^+ is defined as in Equation (36). We will construct an isomorphism

$$\mathcal{V}_1 \;\cong\; X(2,1).$$

This allows us to deduce the desired result using the cancellation property for finitely generated projective modules over $C(\mathbb{T}^2)$.

We first compute for $h_1,\ h_2\ \in\ X(1,1)$ the value of $\langle D(\mathcal{R}^+(h_1), D(\mathcal{R}^+(h_2)\rangle_{C(\mathbb{T}^2)}$:

$$\langle D(\mathcal{R}^+(h_1), D(\mathcal{R}^+(h_2)\rangle_{C(\mathbb{T}^2)}(s,t)$$

$$= \frac{1}{2}\sum_{(m',n')\in[Q^t]^{-1}(\mathbb{Z}^2)} |\sigma^+((\frac{s+t}{2},\frac{-s+t}{2})-(m',n'))|^2 \times \cdots$$

$$\cdots \times\ (\overline{h_1}\cdot h_2)((\frac{s+t}{2},\frac{-s+t}{2})-(m',n')).$$

In the standard fashion, we split this sum into two portions, using the two coset representatives $(0,0)$ and $(\frac{1}{2},\frac{-1}{2})$ for the quotient group $(Q^t)^{-1}(\mathbb{Z}^2)/(\mathbb{Z}^2)$, obtaining

$$\frac{1}{2}\langle\sigma^+\cdot h_1,\sigma^+\cdot h_2\rangle_{C(\mathbb{T}^2)}(\frac{s+t}{2},\frac{-s+t}{2})$$

$$+\ \frac{1}{2}\langle\sigma^+\cdot h_1,\sigma^+\cdot h_2\rangle_{C(\mathbb{T}^2)}((\frac{s+t+1}{2},\frac{-s+t-1}{2})$$

$$=\ \frac{1}{2}[(\overline{h_1}\cdot h_2)(\frac{s+t}{2},\frac{-s+t}{2})+(\overline{h_1}\cdot h_2)(\frac{s+t+1}{2},\frac{-s+t-1}{2})].$$

Define a function $\rho^+ :\ \mathcal{V}_1\ \rightarrow\ X(2,1)$ by

$$\rho^+[D(\mathcal{R}^+(h)](s,t)\ =\ \rho^+[\frac{1}{\sqrt{2}}\sigma^+\circ[Q^t]^{-1}h\circ[Q^t]^{-1}](s,t)$$

$$=\ \tilde{l}(s,t)h(\frac{s+t}{2},\frac{-s+t}{2})+\tilde{l}(s+1,t)h(\frac{s+t+1}{2},\frac{-s+t-1}{2}),$$

where

$$\tilde{l}(s,t)\ =\ \frac{1}{2}e(\frac{st}{4})e(-\frac{s^2+2s}{8})\ e(\frac{t^2+2t}{8}). \tag{38}$$

Note the modulus of $\tilde{l}$ is identically equal to $\frac{1}{2}$. The calculations of the previous paragraph show that in order to have

$$\langle\rho^+[D(\mathcal{R}^+(h_1))],\rho^+[D(\mathcal{R}(h_2))]\rangle_{C(\mathbb{T}^2)}(s,t)=\langle D(\mathcal{R}^+(h_1)),D(\mathcal{R}^+(h_2))\rangle_{C(\mathbb{T}^2)}(s,t)$$

for any choice of $h_1,h_2\in\ X(1,1)$, and

$$\rho^+[D(\mathcal{R}^+(h))](s,t)\ \in\ X(2,1)\ \forall h\in\ X(1,1),$$

it is necessary that

$$\langle\rho^+[D(\mathcal{R}^+(h_1))],\rho^+[D(\mathcal{R}^+(h_2))]\rangle_{C(\mathbb{T}^2)}(s,t)$$

$$=\ \frac{1}{2}[(\overline{h_1}\cdot h_2)(\frac{s+t}{2},\frac{-s+t}{2})+(\overline{h_1}\cdot h_2)(\frac{s+t+1}{2},\frac{-s+t-1}{2})].$$

We first compute using Equation 38 that $\tilde{l}$ satisfies the conditions

$$\tilde{l}(s,t-2) \ = \ e(\frac{s-t}{2})\tilde{l}(s,t), \tag{39}$$

and

$$\tilde{l}(s-2,t) \ = \ e(\frac{s+t}{2})\tilde{l}(s,t) \tag{40}$$

for all $s,\ t \ \in \ \mathbb{R}$. Now note that for every $h \ \in \ X(1,1)$,

$$\begin{aligned}
&\rho^{+}[D(\mathcal{R}^{+}(h)](s-1,t)\\
&= \ \tilde{l}(s-1,t)h(\frac{s-1+t}{2},\frac{-s-1+t+2}{2}) + \tilde{l}(s,t)h(\frac{s+t}{2},\frac{-s+t}{2})\\
&= \ \tilde{l}(s,t)h(\frac{s+t}{2},\frac{-s+t}{2}) + e(-\frac{s-1+t}{2})\tilde{l}(s-1,t)h(\frac{s-1+t}{2},\frac{-s-1+t}{2})\\
&= \ \rho^{+}[D(\mathcal{R}^{+}(h)](s,t),
\end{aligned}$$

by Equation (40). Also, computing further, we see that

$$\begin{aligned}
&\rho^{+}[D(\mathcal{R}^{+}(h)](s,t-2)\\
&= \ \tilde{l}(s,t-2)h(\frac{s+t-2}{2},\frac{-s+t-2}{2}) + \tilde{l}(s+1,t-2)h(\frac{s+t-1}{2},\frac{-s+t-3}{2})\\
&= \ e(\frac{s+t}{2})[\tilde{l}(s,t-2)h(\frac{s+t}{2},\frac{-s+t}{2}) + e(-1/2)\tilde{l}(s+1,t-2)h(\frac{s+t+1}{2},\frac{-s+t-1}{2})],
\end{aligned}$$

which, by Equations (39) and (40),

$$= \ e(s)\rho^{+}[D(\mathcal{R}^{+}(h)](s,t).$$

Hence,

$$\rho^{+}[D(\mathcal{R}^{+}(h)] \ \in \ X(2,1),\ \forall\, h \ \in \ X(1,1).$$

We must finally show that the inner product identity

$$\langle \rho^{+}[D(\mathcal{R}^{+}(h_1))], \rho^{+}[D(\mathcal{R}^{+}(h_2))]\rangle_{C(\mathbb{T}^2)}(s,t) = \ \langle D(\mathcal{R}^{+}(h_1)), D(\mathcal{R}^{+}(h_2))\rangle_{C(\mathbb{T}^2)}(s,t)$$

is satisfied. Using the definition of $\rho^{+}[D(\mathcal{R}^{+}(h_1))]$ and the inner product on $X(2,1)$, one computes:

$$\begin{aligned}
&\langle \rho^{+}[D(\mathcal{R}^{+}(h_1))], \rho^{+}[D(\mathcal{R}^{+}(h_2))]\rangle_{C(\mathbb{T}^2)}(s,t)\\
&= \ 2[\frac{1}{2}]^2[(\overline{h_1}\cdot h_2)(\frac{s+t}{2},\frac{-s+t}{2}) \ + \ (\overline{h_1}\cdot h_2)(\frac{s+t-1}{2},\frac{-s+t-1}{2})]\\
&\quad + M(s,t)\overline{h_1(\frac{s+t}{2},\frac{-s+t}{2})}h_2(\frac{s+t-1}{2},\frac{-s+t-1}{2})\\
&\quad + \overline{M(s,t)}\,\overline{h_1(\frac{s+t-1}{2},\frac{-s+t-1}{2})}h_2(\frac{s+t}{2},\frac{-s+t}{2}),
\end{aligned}$$

for

$$M(s,t) \;=\; \overline{\tilde{l}(s,t)}\tilde{l}(s+1,t) \;+\; e(-\frac{s+t}{2})\overline{\tilde{l}(s+1,t-1)}\tilde{l}(s,t-1). \qquad (41)$$

We now verify the identity

$$\overline{M(s,t)} \;\equiv\; M(s,t) \;\equiv\; 0, \qquad (42)$$

and the proof of the equality of the inner products will follow, since we have already shown that

$$\langle D(\mathcal{R}^+(h_1)), D(\mathcal{R}^+(h_2))\rangle_{C(\mathbb{T}^2)}(s,t)$$

$$= \frac{1}{2}[(\overline{h_1}\cdot h_2)(\frac{s+t}{2},\frac{-s+t}{2}) + (\overline{h_1}\cdot h_2)(\frac{s+t+1}{2},\frac{-s+t-1}{2})].$$

Using the definition of M given in Equation (41) and the definition of $\tilde{l}$ given in Equation 38, we calculate

$$\overline{\tilde{l}(s+1,t)}\tilde{l}(s,t) \;=\; \frac{1}{4}e(\frac{s+t}{4})e(\frac{3}{8}),$$

as $\tilde{l}(s,t)$ has constant modulus $\frac{1}{2}$. A similar calculation yields

$$e(\frac{s+t}{2})\overline{\tilde{l}(s,t-1)}\tilde{l}(s+1,t-1) \;=\; \frac{1}{4}e(\frac{s+t}{4})e(-\frac{1}{8}).$$

Hence

$$\overline{M(s,t)} = \overline{\tilde{l}(s+1,t)}\tilde{l}(s,t) \;+\; e(\frac{s+t}{2})\overline{\tilde{l}(s,t-1)}\tilde{l}(s+1,t-1)$$

$$= \frac{1}{4}e(\frac{s+t}{4})e(\frac{3}{8}) \;+\; \frac{1}{4}e(\frac{s+t}{4})e(-\frac{1}{8}) \;=\; 0,$$

and we have verified Equation (42), as we desired to show. The only thing left to do is to show that the map ρ^+ from $\mathcal{V}_1$ to $X(2,1)$ is surjective, but this is a calculation similar to that done in Section 3, which we leave to the reader. As mentioned in the beginning of the proof, the fact that

$$\mathcal{W}_0 \;\cong\; X(1,0) \;\cong\; C(\mathbb{T}^2)$$

follows immediately from the isomorphism between $\mathcal{V}_1$ and $X(2,1)$, and the cancellation property for finitely generated projective $C(\mathbb{T}^2)$-modules.

Corollary 20. *Let* $\mathcal{V}_0 \;\oplus\; \oplus_{i=0}^{\infty}\mathcal{W}_i$ *be decomposition of* Ξ *corresponding to the projective multi-resolution analysis related to the quincunx matrix* Q *given above. Then for each* $i \in \mathbb{N}\cup\{0\}$, $\mathcal{W}_i$ *is a free* $C(\mathbb{T}^2)$ *module of rank* 2^i.

Proof. This follows immediately from Theorem 19 and Theorem 5.1 of [17].

An important fact to note in the proofs of both Theorems 18 and 16, as noted in the discussion given in Section 5 of [20], is that in order to construct projective multi-resolution analyses for an integer dilation matrix A, we need the existence of ordinary low-pass filter functions m_0 for the corresponding dilation matrix A such that:

(i) m_0 satisfies the usual wavelet conditions, i.e. m_0 is continuously differentiable at 0 with $m_0(0,0) = 1$,

$$\sum_{j=0}^{d-1} |m_0((s,t) + \mathbf{v_j})|^2 = 1,$$

where $\{\mathbf{v_j}\}_{j=0}^{d-1}$ is a collection of coset representatives in $[A^t]^{-1}(\mathbb{Z}^2)$ for the d-element group $[A^t]^{-1}(\mathbb{Z}^2)/\mathbb{Z}^2$, and m_0 satisfies Cohen's condition;

(ii) m_0 is sufficiently regular that $\prod_{j=1}^{\infty} m_0([A^t]^{-j}(s,t)) = \widehat{\phi}(s,t)$ lies in $\varXi$, i.e. $\widehat{\phi}$ needs to have sufficient decay properties;

(iii) m_0 vanishes along some sort of line or curve in $\mathbb{R}^2/\mathbb{Z}^2$, so that we can multiply it by a jump function that has a jump discontinuity along that very same curve to obtain a new continuous function $\widetilde{m}_0$ satisfying certain necessary functional relations that will ensure $D_{A^t}(\mathcal{R}(X(q,a))) \subset \mathcal{R}(X(q,a))$. Here, as previously, $\mathcal{R}$ is multiplication by $\sigma \in \varXi$ constructed from $\widetilde{m}_0$, so the method to date has been to find $\widetilde{m}_0$ so that $\widetilde{m}_0(s,t) \cdot h \circ A^t(s,t) \in X(q,a)$, for all $h \in X(q,a)$.

Once having obtained an ordinary low-pass filter of this type, the construction of the embedding of the projective $C(\mathbb{T}^2)$-module into $\varXi$ as the core projective module $\mathcal{V}_0$ is not too difficult. The identification of the isomorphism type of $\mathcal{V}_1 = D(\mathcal{V}_0)$ as carried out in the proofs of Theorems 17 and 19 at this point seems to require various delicate maneuvers in calculation, which vary somewhat depending on the matrix being considered.

5 Module frames and ordinary frames in $L^2(\mathbb{R}^2)$

We now discuss how to use the results of the previous sections to construct normalized tight frames (Parseval frames) for $L^2(\mathbb{R}^2)$. We first review a definition due in its greatest generality to M. Frank and D. Larson ([9]).

Definition 21. *Let $\mathcal{A}$ be a unital C^*-algebra, let $\mathcal{V}$ be a finitely generated projective A-submodule which is a right Hilbert C^*-module over $\mathcal{A}$. Then the set $\{\phi_1, \phi_2, \cdots, \phi_m\} \subseteq \mathcal{V}$ is called a module-frame for $\mathcal{V}$ if for all $\zeta \in \mathcal{V}$, the following reconstruction formula is satisfied.*

$$\zeta = \sum_{1 \le j \le m} \phi_j \langle \phi_j, \zeta \rangle_{\mathcal{A}}.$$

In the case where $\mathcal{A} = C(\mathbb{T}^2)$ or even $C(\mathbb{T}^n)$ for $n > 2$, it is quite easy to come up with some explicit constructions of module frames, given the explicit form of the projective module used as initial module. They are a replacement for ordinary frames in the Hilbert space setting. For completeness, we give the definition of Hilbert space frames as well:

Definition 22. *Recall that a sequence $\{f_k : k \in \mathbb{N}\}$ of elements in a Hilbert space $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there are real constants $C, D > 0$ such that*

$$C \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \ \leq \ \|f\|^2 \ \leq \ D \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2$$

for every $f \in \mathcal{H}$ where here the inner product is Hilbert space inner product. If $C = D = 1$, the frame is said to be a normalized tight frame or Parseval frame for $\mathcal{H}$.

Parseval frames $\{f_k : k \in \mathbb{N}\}$ for $\mathcal{H}$ give perfect reconstruction in the Hilbert space setting:

$$f \ = \ \sum_{k=1}^{\infty} \langle f, f_k \rangle_{\mathcal{H}} f_k.$$

Example 23. We construct a module frame for $X(1,a)$: Let m_0 be a continuous normalized low-pass filter defined on $\mathbb{R}$ for dilation by 2, so that

$$m_0(0) \ = \ 1, \text{ and}$$

$$\sum_{j=0}^{1} |m_0(t + \frac{j}{2})|^2 \ = \ 1, \text{ so that } m_0(\frac{1}{2}) = 0.$$

Define

$$h_0(s,t) \ = m_0(t) J_0(s,t), \tag{43}$$

and

$$h_1(s,t) = m_0(t + \frac{1}{2}) J_1(s,t), \tag{44}$$

where

$$J_0(s,t) = \ e(-nas), \ t \ \in \ [n - \frac{1}{2}, n + \frac{1}{2}), \ n \ \in \ \mathbb{Z},$$

and

$$J_1(s,t) = \ e(-nas), \ t \ \in \ [n, n+1), \ n \ \in \ \mathbb{Z}.$$

Then $h_0, \ h_1 \ \in \ X(1,a)$, and one can calculate that they give a $C(\mathbb{T}^2)$-module frame for $X(1,a)$. The idea of strong Morita equivalence bimodules is essential to this calculation.

We note that the jump functions J_0 and J_1 are just relabeled forms of the jump functions $J_{q,a,\beta} : \mathbb{R}^2 \to \mathbb{C}$ reviewed in Equation 16. Using this notation, we see that the functions J_0 and J_1 of Example 23 are $J_{1,-a,-1/2}$ and $J_{1,-a,0}$, respectively.

By using the above construction and the isomorphism $X(q,a) \cong X(1,a) \oplus X(q-1,0)$ for positive integers $q > 1$, we note that for the non-free finitely generated projective modules $X(q,a)$ of dimension q described earlier, the number of elements in any module frame will be at least $q+1$ (this is an illustration of redundancy inherent in frame theory). Hence, if $a \neq 0$, the least number of elements in the module frame for the initial modules $\mathcal{V}_0$ that are isomorphic to $X(q,a)$ constructed in earlier sections is $q+1$.

Example 24. We now use Example 23 and Theorem 12 to construct a first a module frame, and then a Parseval frame, for the initial module $\mathcal{V}_0$ and its closure in $L^2(\mathbb{R}^2)$, respectively, corresponding to the projective multi-resolution analysis in the special case where $q = 1$ and the dilation matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Recall in this case our module map is given by

$$\mathcal{R}(h)(s,t) \;=\; \sigma(s,t)h(s,t),\; h \in X(1,a),$$

for

$$\sigma(s,t) \;=\; \prod_{j=1}^{\infty}[\widetilde{m}_0(\frac{s}{2^j},\frac{t}{2^j})],$$

where $\widetilde{m}_0 \in X(1,-3a)$ and in addition $\widetilde{m}_0$ satisfies the normalized low-pass filter equations corresponding to the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Indeed the proof of Theorem 5 of [20] shows how we can take

$$\widetilde{m}_0(s,t) \;=\; m_0(s)m_0(t)J_{1,-3,-1/2}(s,t),$$

where m_0 is a continuous low-pass filter for dilation by 2 defined on $\mathbb{R}/\mathbb{Z}$ such that the corresponding function σ lies in Ξ. It follows that the two element $C(\mathbb{T}^2)$-module frame for $\mathcal{V}_0$ in this case will be

$$\{\mathcal{R}(h_0) :=\; m_0(t)J_{1,-a,-1/2}(s,t)\sigma(s,t),\; \mathcal{R}(h_1) :=\; m_0(t+\frac{1}{2})J_{1,-a,0}(s,t)\sigma(s,t)\}. \tag{45}$$

By Proposition 12 of [20],

$$\{\mathcal{R}(h_0)\cdot\sigma(s,t)\cdot e(-m\cdot(s,t)) : m \in \mathbb{Z}^2\} \cup \{\mathcal{R}(h_1)\cdot\sigma(s,t)\cdot e(-m\cdot(s,t)) : m \in \mathbb{Z}^2\}$$

is a normalized tight frame for $\overline{\mathcal{V}_0} \;=\; V_0 \subset L^2(\mathbb{R}^2)$.

A result of Frank and Larson tell us that if $\mathcal{J}$ is a countably generated right Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$, then $\mathcal{J}$ will have a **countable** standard module-frame, from which one can deduce that there will exist a countable set $\{\phi_1, \phi_2, \cdots,\} \subseteq \mathcal{J}$ such that

$$\zeta = \sum_{k \in \mathbb{N}} \phi_k \langle \phi_k, \zeta \rangle_{\mathcal{A}}. \tag{46}$$

So in particular, $\varXi$ should have a module frame over $C(\mathbb{T}^2)$. Given a projective multi-resolution analysis $\{\mathcal{V}_j\}$ in $\varXi$ for dilation by A, explicit formulas for module frames for $\varXi$ can be obtained as follows.
Step 1: We first prove that

$$\varXi = \mathcal{V}_0 \oplus_{j\geq 0}^{\infty} \mathcal{W}_j \text{ (topological direct sum).}$$

We remark here that the topological direct sum

$$\mathcal{V}_0 \oplus_{j\geq 0}^{\infty} \mathcal{W}_j$$

can be identified with

$$\{v + \{w_j\}_{j\geq 0} : w_j \in \mathcal{W}_j, \text{and} \sum_{j\geq 0} < w_j, w_j >_{C(\mathbb{T}^2)} \text{ converges}\}.$$

Step 2: Find a module frame for the initial module $\mathcal{V}_0$ and each of the modules $\mathcal{W}_j$, by using an isomorphism the modules $\mathcal{W}_j$ and $X(q_j, a_j)$, and finding a family of module frames for $X(q_j, a_j)$ that can be explicitly written down for each j.

The benefits of this method are that by using different projective multi-resolution analyses for $\varXi$, we can construct different module frames for $\varXi$. Then, using these module frames for $\varXi$, we can construct ordinary normalized tight frames in the Hilbert space setting for $L^2(\mathbb{R}^2)$, as in Example 45.

Using the fact that $\varXi$ is dense in $L^2(\mathbb{R}^2)$ in the L^2 norm, and that each Hilbert module $\mathcal{V}_0$ and $\mathcal{W}_j$ is dense in the closed subspaces V and W_j of $L^2(\mathbb{R}^2)$, respectively, arising as in ordinary multi-resolution analysis theory, it is possible to use families of module frame for $\mathcal{V}_0$ and each $\mathcal{W}_j$ to construct a normalized tight frame for V_0 and each W_j. Since

$$L^2(\mathbb{R}^2) = V_0 \oplus [\oplus_{j=0}^{\infty} W_j], \text{ (Hilbert space sum)}$$

the union of the normalized tight frames for V_0 and each W_j as j runs over $\mathbb{N} \cup \{0\}$ gives a normalized tight frame for $L^2(\mathbb{R}^2)$.

Example 25. If $\{\psi_1, \psi_2, \cdots, \psi_d\}$ is a module frame for the wavelet module $\mathcal{W}_0$, it is possible to show that

$$\cup_{k=1}^{d} \{\psi_k \cdot e(-m \cdot (s,t)) \ : \ m \in \mathbb{Z}^2\}$$

is a Parseval frame for $\overline{\mathcal{W}_0} = W_0 \subseteq L^2(\mathbb{R}^2)$; see [20] Proposition 12 for details.

6 Conclusion and open problems

The methods we have used so far may seem fairly specialized. The question then arises as to how to construct examples beyond the special cases of diagonal matrices with integer entries, their conjugates by elements of $SL(n,\mathbb{Z})$, and the special matrices P and Q discussed above. We hope that this article indicates a general method at the end of Section 4 for the 2×2 case. Indeed, the construction of an **ordinary** multi-resolution analysis for a general $n \times n$ dilation matrix with integer entries was first accomplished in 2001 by M. Bownik in [4]; his construction would presumably prove useful as the scaling functions and wavelets he constructs have arbitrary degrees of smoothness. A special case of our construction gives the Fourier transform of a dense subspace of certain ordinary multi-resolution analyses, with this dense subspace lying inside the Hilbert $C(\mathbb{T}^2)$ module Ξ mentioned in the first section.

In the case where $n = 2$, Bownik and Speegle have recently shown that for any 2×2 dilation matrix it is possible to construct a scaling function whose Fourier transform is smooth and has compact support in $\mathbb{R}^2$, hence will lie in the Hilbert $C(\mathbb{T}^2)$-module Ξ. For $C(\mathbb{T}^n)$-modules with $n \geq 3$, the problem of constructing an ordinary multi-resolution analysis for an arbitrary dilation matrix such that the initial space V_0 has a dense subspace invariant under translation by $\mathbb{Z}^n$ whose Fourier transform lies inside Ξ remains an open one.

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