1. *(12 points)*

   Let $(X, \mathcal{T})$ be a compact Hausdorff space, and suppose that $\mathcal{F} \subseteq C(X)$ is a collection of functions that is equicontinuous and pointwise bounded (i.e. for each $x \in X$, the set $\{f(x) : f \in \mathcal{F}\}$ is bounded). Prove that $\mathcal{F}$ is uniformly bounded, i.e. prove that there exists $M > 0$ such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and for all $x \in X$.

   Solution: The conditions of the Arzela-Ascoli Theorem are satisfied, and therefore the family $\mathcal{F}$ is totally bounded. So, for any $\epsilon > 0$, the family $\mathcal{F}$ can be covered by a finite number of ϵ balls. Take $\epsilon = 1$ and find $N \in \mathbb{N}$ and $\{f_1, f_2, \cdots, f_N\} \subset \mathcal{F}$ such that

   $$\mathcal{F} \subset \cup_{i=1}^{N} B(f_i, 1).$$

   Therefore for all $f \in \mathcal{F}$ there exists some i_0 with

   $$\|f - f_{i_0}\|_\infty < 1.$$

   But then

   $$\|f\| - \|f_{i_0}\| \leq \|f - f_{i_0}\|_\infty < 1,$$

   so that

   $$\|f\| < 1 + \|f_{i_0}\|.$$

   Let

   $$M_i = 1 + \|f_i\| = 1 + \sup_{x \in X} |f_i(x)|, \; 1 \leq i \leq N,$$

   and set $M = \max\{M_1, M_2, \cdots, M_N\}$. We then have for $f \in \mathcal{F}$ chosen as above

   $$\sup_{x \in X} |f(x)| < 1 + M_{i_0} \leq M,$$

   and we see that $\mathcal{F}$ is uniformly bounded.

2. *(13 points)* Let $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$. Recall that $\mathbb{T}$ given the relative topology induced from the standard topology on $\mathbb{C}$ is a compact metric space. Consider the set

$$\mathcal{L} = \cup_{M=0}^{\infty} \cup_{N=0}^{\infty} \{P(z) = \sum_{j=-M}^{N} a_j z^j : a_j \in \mathbb{C}, \ \forall j \in \mathbb{Z}\} \subset C(\mathbb{T}).$$

Prove that $\mathcal{L}$ is dense in $C(\mathbb{T})$ in the uniform norm.

Solution: Here we use the Stone-Weierstrass Theorem (complex version). The set $\mathcal{L}$ of "Laurent polynomials" on the circle is easily seen to be an algebra (closed under pointwise addition, scalar multiplication, and pointwise multiplication of two polynomials). It also contains the complex constant functions, because choosing $a \in \mathbb{C}$, then $p(z) = az^0 = a, \ \forall z \in \mathbb{T}$ is an element of $\mathcal{L}$. It also separates points, because if we are given $x, y \in \mathbb{T}$ with $x \neq y$, then if we consider $I(z) = z^1 = z$, we have $I \in \mathcal{L}$ and $I(x) = x \neq y = I(y)$. Finally $\mathcal{L}$ is closed under complex conjugation, because if we are given $P(z) = \sum_{j=-M}^{N} a_j z^j \in \mathcal{L}$, then

$$\overline{P(z)} = \overline{\sum_{j=-M}^{N} a_j z^j}$$

$$= \sum_{j=-M}^{N} \overline{a_j z^j} = \sum_{j=-M}^{N} \overline{a_j} z^{-j}$$

$$= \sum_{j=-N}^{M} \overline{a_{-j}} z^j \ \in \mathcal{L}.$$

Therefore the closure of $\mathcal{L}$ in $C(\mathbb{T})$ in the uniform norm satisfies all these same properties, so that by the Stone-Weierstrass Theorem (complex version), it must be equal to all of $C(\mathbb{T})$, so that $\mathcal{L}$ is dense in $C(\mathbb{T})$, as desired.

3. *(13 points)* Let $\mathcal{X}$ be a normed vector space over $\mathbb{C}$. Let $\mathcal{X}^* = \mathcal{L}(\mathcal{X}, \mathbb{C})$ be the space of all bounded linear functionals from $\mathcal{X}$ to $\mathbb{C}$. Assuming that $\mathcal{X}^*$ is a normed vector space under the operator norm, show that it is complete, i.e. show that if $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{X}^*$ in the operator norm, then it is convergent to some element ϕ_0 of $\mathcal{X}^*$.

Solution: Let $\{\phi_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the operator norm on $\mathcal{X}^*$, i.e. the standard norm for linear functionals. We now let $x \in \mathcal{X}$ and consider the sequence of complex numbers $\{\phi_n(x)\}_{n=1}^{\infty}$. Since $\{\phi_n\}_{n=1}^{\infty}$ is Cauchy in $\mathcal{X}^*$, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$,

$$\|\phi_n - \phi_m\| < \varepsilon.$$

It follows that whenever $n > m \geq N$,

$$|\phi_n(x) - \phi_m(x)| = \ |(\phi_n - \phi_m)(x)| \leq \|\phi_n - \phi_m\| \cdot \|x\| < \epsilon\|x\|.$$

But this means that the sequence of complex numbers $\{\phi_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$, so has a limit, which we denote by $\phi_0(x)$. It is easy to check that ϕ_0 is a linear functional and I leave that to you. I check that ϕ_0 is bounded, first of all. We have for $n > m = N$ and for $x \in \mathcal{X}$ with $\|x\| \leq 1$,

$$|\phi_n(x)| - |\phi_N(x)| \leq |(\phi_n - \phi_N)(x)| \leq \|\phi_n - \phi_N\| \cdot \|x\| \leq \|\phi_n - \phi_N\| < \varepsilon.$$

Therefore, for all $n \geq N$, and for all $x \in \mathcal{X}$ with $\|x\| \leq 1$,

$$|\phi_n(x)| < \varepsilon + |\phi_N(x)| \leq \varepsilon + \|\phi_N\| \cdot \|x\| \leq \varepsilon + \|\phi_N\|.$$

Letting $n \to \infty$ we get for $x \in \mathcal{X}$ with $\|x\| \leq 1$:

$$|\phi_0(x)| = |\lim_{n\to\infty} \phi_n(x)| = \lim_{n\to\infty} |\phi_n(x)| \leq \varepsilon + \|\phi_N\| < \infty,$$

so that $\|\phi_0\| = \sup\{|\phi_0(x)| : x \in \mathcal{X},\ \|x\| \leq 1\}$ is finite and $\phi_0 \in \mathcal{X}^*$. Finally, we have seen that for all $n > m \geq N$,, we have seen that for all $x \in \mathcal{X}$ with $\|x\| \leq 1$,

$$|\phi_n(x) - \phi_m(x)| = \ |(\phi_n - \phi_m)(x)| \leq \|\phi_n - \phi_m\| \cdot \|x\| < \epsilon.$$

Letting $n \to \infty$, and fixing $m \geq N$, we obtain that for all $x \in \mathcal{X}$ with $\|x\| \leq 1$,

$$|\phi_0(x) - \phi_m(x)| = \ |\lim_{n\to\infty}(\phi_n - \phi_m)(x)| = \lim_{n\to\infty} |\phi_n(x) - \phi_m(x)| \leq \epsilon.$$

But this means that whenever $m \geq N$, we have $\|\phi_0 - \phi_m\| \leq \epsilon$, so that

$$\lim_{m\to\infty} \phi_m = \phi_0$$

in $\mathcal{X}^*$ with respect to the linear functional norm, and $\mathcal{X}^*$ is complete.

4. *(12 points)*

(a) Prove that $\mathbb{Q}$ is a subset of the first category (a meager subset) of $\mathbb{R}$, where $\mathbb{R}$ is given its usual metric. Be sure to justify your reasoning.

Solution: $\mathbb{Q}$ is countable so that we can enumerate it in some fashion:

$$\mathbb{Q} \;=\; \{q_n : n \in \mathbb{N}\} = \cup_{n=1}^{\infty}\{q_n\}.$$

But each point set $\{q_n\}$ is closed in $\mathbb{R}$ and has empty interior, i.e. each point set $\{q_n\}$ is nowhere dense in $\mathbb{R}$. Therefore, $\mathbb{Q}$ can be expressed as a countable union of nowhere dense sets, so it is a subset of $\mathbb{Q}$ of the first category.

(b) Using part (a) and the Baire Category Theorem, prove that $\mathbb{R} \setminus \mathbb{Q}$ is a subset of the second category of $\mathbb{R}$.

Solution: Suppose, by way of contradiction, that $\mathbb{R} \setminus \mathbb{Q}$ is a set of the first category in $\mathbb{R}$. Write $\mathbb{R} \setminus \mathbb{Q} = \cup_{m=1}^{\infty} B_m$ where each B_m is nowhere dense in $\mathbb{R}$. Then

$$\mathbb{R} \;=\; \mathbb{Q} \cup [\mathbb{R} \setminus \mathbb{Q}] \;=\; \cup_{n=1}^{\infty}\{q_n\} \cup \cup_{m=1}^{\infty} B_m,$$

so that $\mathbb{R}$ is a countable union of nowhere dense sets and is therefore a set of the first category in itself. Since $\mathbb{R}$ is a complete metric space, this last statement contradicts the Baire Category Theorem, and therefore what we assumed is incorrect, and $\mathbb{R} \setminus \mathbb{Q}$ must be a set of the second category in $\mathbb{R}$.