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Abstract

Benacerraf's Dilemma (BD), as formulated by Paul Benacerraf in "Mathematical Truth," is about the apparent impossibility of reconciling a "standard" (i.e., classical Platonic) semantics of mathematics with a "reasonable" (i.e., causal, spatiotemporal) epistemology of cognizing true statements. In this paper I spell out a new solution to BD. I call this new solution a *positive Kantian phenomenological solution* for three reasons: (1) It accepts Benacerraf's preliminary philosophical assumptions about the nature of semantics and knowledge, as well as all the basic steps of BD, and then shows how we can, consistently with those very assumptions and premises, still reject the skeptical conclusion of BD and also adequately explain mathematical knowledge. (2) The standard semantics of mathematically necessary truth that I offer is based on Kant's philosophy of arithmetic, as interpreted by Charles Parsons and by me. (3) The reasonable epistemology of mathematical knowledge that I offer is based on the phenomenology of logical and mathematical self-evidence developed by early Husserl in *Logical Investigations* and by early Wittgenstein in *Tractatus Logico-Philosophicus*.

CHAPTER VIII

MATHEMATICAL TRUTH REGAINED

Robert Hanna

Abstract. Benacerraf's Dilemma (BD), as formulated by Paul Benacerraf in "Mathematical Truth," is about the apparent impossibility of reconciling a "standard" (i.e., classical Platonic) semantics of mathematics with a "reasonable" (i.e., causal, spatiotemporal) epistemology of cognizing true statements. In this paper I spell out a new solution to BD. I call this new solution a *positive Kantian phenomenological solution* for three reasons: (1) It accepts Benacerraf's preliminary philosophical assumptions about the nature of semantics and knowledge, as well as all the basic steps of BD, and then shows how we can, consistently with those very assumptions and premises, still reject the skeptical conclusion of BD and also adequately explain mathematical knowledge. (2) The standard semantics of mathematically necessary truth that I offer is based on Kant's philosophy of arithmetic, as interpreted by Charles Parsons and by me. (3) The reasonable epistemology of mathematical knowledge that I offer is based on the phenomenology of logical and mathematical self-evidence developed by early Husserl in *Logical Investigations* and by early Wittgenstein in *Tractatus Logico-Philosophicus*.

I who erewhile the happy garden sung,
By one man's disobedience lost, now sing
Recovered Paradise to all mankind,
By one man's firm obedience fully tried
Through all temptation, and the Tempter foiled
In all his wiles, defeated and repulsed,
And Eden raised in the waste wilderness.

–J. Milton[1]

Pure intuition as Kant understood it was evidently supposed somehow to get us across the divide between the fuzzy *Lebenswelt* with its everyday objects and the sharp, precise realm of the mathematical, in terms of which mathematical conceptions of the physical world are developed.

–C. Parsons[2]

[1] (Milton, 1953b, 495, book I, lines 1–7)
[2] (Parsons 2008, 166)

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The epistemologically pregnant sense of self-evidence (*Evidenz*) . . . gives to an intention, e.g., the intention of judgment, the absolute fullness of content, the fullness of the object itself. The object is not merely meant, but in the strictest sense *given*, and given as it is meant, and made one with our meaning-reference. . .. It is said of every percept that it grasps its object directly, or grasps this object *itself*. But this direct grasping has a different sense and character according as we are concerned with a percept in the narrower or wider sense, or according as the directly grasped object is *sensible* or *categorial*. Or otherwise put, according as it is a *real* or *ideal* object.

–E. Husserl[3]

Self-evidence (*die Einleuchten*), of which Russell has said so much, can only be discarded in logic by language itself preventing every logical mistake. That logic is a priori consists in the fact that we *cannot* think illogically.

–L. Wittgenstein[4]

I. Introduction

Benacerraf's Dilemma, or BD, as originally formulated by Paul Benacerraf in 1973 (Benacerraf 1973, 672–673), is about the apparent impossibility of reconciling a *standard, uniform* semantics of natural language with a *reasonable* epistemology of cognizing true statements, when the relevant kind of true statement to be semantically explained is mathematical truth and the relevant kind of cognition to be epistemologically explained is mathematical knowledge. A "standard, uniform" semantics in Benacerraf's terminology is a Tarskian satisfaction-theoretic and model-theoretic semantics applying across natural language as a whole. This semantics, together with some natural assumptions about standard mathematical linguistic practices, very plausibly, smoothly, and jointly yield classical Platonism about mathematics. And a "reasonable" epistemology is an epistemology that ties a human linguistic knower causally to the known objects themselves. This epistemology very plausibly and smoothly yields the *denial* of classical Platonism about mathematics. Hence BD.

[3](LI, pp. 765 and 787, texts combined)
[4](Wittgenstein, 1981, prop. 5.4731, p. 129)

In this paper I will spell out a new solution to BD. I call this new solution a *positive Kantian phenomenological solution* for three reasons:

(1) It accepts Benacerraf's preliminary philosophical assumptions about the nature of semantics and knowledge, as well as all the basic steps of BD, and then shows how we can, consistently with those very assumptions and premises, *still* reject the skeptical conclusion of BD and *also* adequately explain mathematical knowledge.
(2) The standard semantics of mathematically necessary truth that I offer is based on Kant's philosophy of arithmetic, as interpreted by Charles Parsons and by me. (See Parsons 1983; Hanna 2002, 2006a, Chapter 6.)
(3) The reasonable epistemology of mathematical knowledge that I offer is based on the phenomenology of logical and mathematical self-evidence developed by early Husserl in *Logical Investigations* and by early Wittgenstein in *Tractatus Logico-Philosophicus*.

More precisely, however, what I will argue is that we can solve BD in three stages:

First, I accept Benacerraf's preliminary philosophical intuitions about the nature of semantics and knowledge, as well as all the basic premises of BD.

Second, I hold that mathematical truth is adequately explained by accepting the following three claims:

(1) that the natural numbers are essentially positions or roles in the mathematical natural number structure provided by Peano arithmetic,
(2) that the mathematical natural number structure provided by Peano arithmetic is abstract only in the sense that it is *transcendentally ideal*, which is to say that this structure is identical to the formal structure of time insofar as we consciously represent it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human understanding of Peano arithmetic, and
(3) that in our actual world, the unique, intended model of the natural number structure provided by Peano arithmetic is just the set of

manifestly real directly perceivable spatiotemporal material objects—the natural inhabitants of Parsons's "fuzzy *Lebenswelt* with its everyday objects"—insofar as *they are the role players of the Peano-arithmetic-specified natural number roles* in the abstract formal structure of time as we consciously represent it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human understanding of Peano arithmetic.

Third, I hold that mathematical knowledge is grounded on

(1) a rational human agent's *mental-model-manipulating* abilities, which are innately specified in the agent's mind and also inherently present, as necessary ingredients, in human sense perception, and which entail her self-conscious cognition of phenomenologically self-evident formal non-conceptual structures of human sense perception, together with
(2) a rational human agent's *logic-and-language-constructing* abilities, which are innately specified in the agent's mind and also inherently present, as necessary ingredients, in human empirical conceptualizing and perceptual judgment, and which entail her self-conscious cognition of phenomenologically self-evident formal conceptual contents and specific patterns of logical inference in classical or non-classical logics.

The second and third stages of this argument respectively invoke what I call *Kantian Structuralism* about the nature of numbers and mathematical truth, and also what I call *the Husserl-Wittgenstein Theory of Logical and Mathematical Phenomenological Self-Evidence*, or the HW Theory, about the nature of logical and mathematical a priori knowledge. As the labels clearly indicate, this part of the argument has historical foundations in the work of Kant, early Husserl, and early Wittgenstein. At the same time, however, Kantian Structuralism and the HW Theory are intended to be fully rationally defensible on their own merits.

Now Milton's *Paradise Lost* and *Paradise Regained*, as I read them, are about the necessary transition from the impossibly super-human conception of moral virtue embodied in pre-lapsarian Adam and Eve, and our consequent tragic Fall and expulsion from the Garden of Eden, towards a fully realistic knowledge of our own moral limits and of our inescapably finite, mortal role on this desperately imperfect Earth. Correspondingly, the philosophical story I am telling in this paper is about the necessary philosophical transition from the impossibly superhuman conception of mathematical truth and knowledge offered by classical Platonism, and our consequent tragic Fall and collapse into BD, towards a fully realistic and also inescapably anthropocentric conception of mathematical truth and knowledge, but without either finitism or nominalism—*real mathematics for humans*. So if my argument is sound, then the result will be, in effect, a semantic and epistemic Paradise Regained—with Kantian, Husserlian, and Wittgensteinian bells on.

II. Benacerraf's Dilemma and Some Negative or Skeptical Solutions

Here is Benacerraf's own formulation of BD:

> As an account of our knowledge about medium-sized objects, in the present, this is along the right lines. [A reasonable epistemology] will involve, causally, some direct reference to the facts known, and, through that, reference to those objects themselves. . .. [C]ombining *this* view of knowledge with the "standard" view of mathematical truth makes it difficult to see how mathematical knowledge is possible. If, for example, numbers are the kinds of entities they are normally taken to be, then the connection between the truth conditions for the statements of number theory and any relevant events connected with the people who are supposed to have knowledge cannot be made out. (Benacerraf 1973, 672–673)

And here is my rational reconstruction of that argument:

(1) Natural language requires a standard, uniform semantics. (Preliminary assumption I.)
(2) A reasonable epistemology of cognizing true statements should be modeled on sense perception. (Preliminary assumption II.)
(3) Mathematical knowledge in a classical sense (i.e., as a priori knowledge) exists as a feature of standard mathematical linguistic practices, so mathematical truth in a classical sense (i.e., as necessary truth) also exists as a feature of those standard practices.

(4) Given (1) and (3), our standard, uniform semantics of natural language, as applied to mathematical truths, commits us to a truth-making ontology of abstract mathematical objects and also to the non-empirical knowability of true mathematical statements.
(5) On the one hand, given (2), that fact that a reasonable epistemology of cognizing true statements should be modeled on sense perception entails that knowledge involves causally efficacious, contact-involving or efficient, directly referential, non-inferential, and spatiotemporal relations between human linguistic knowers and the known objects themselves.
(6) But on the other hand, given (4), and since all abstract objects are causally isolated and inert, it then follows that all abstract mathematical objects are causally isolated and inert.
(7) So if we accept all of (1)–(6), then mathematical knowledge in the classical sense is both possible and impossible, which is absurd.

I will say that any proposed solution to BD is *negative* or *skeptical* if it rejects either of Benacerraf's preliminary philosophical assumptions about a standard uniform semantics and a reasonable epistemology or else rejects one or more of steps (3) to (6). Then there are at least six different categories of possible negative or skeptical solutions to BD. The first two categories I will call *pre-emptive* negative or skeptical solutions, since they consist in pre-emptively rejecting at least one of the two preliminary assumptions.

1. Pre-emptive Negative or Skeptical Solutions

(1) *Reject the preliminary assumption (I) that natural language requires a standard, uniform semantics.*

This is in turn entails either

(1.1) rejecting Tarskian semanticsor
(1.2) accepting a multiform semantics of natural language.

(2) *Reject the preliminary assumption (II) that a reasonable epistemology of cognizing true statements should be modeled on sense perception.* (See, e.g., Katz 1995)

This in turn entails either

(2.1) modelling the epistemology of cognizing true statements on conceptual reasoning or concept-possession (see, e.g., Hale and Wright 2002),
(2.2) modelling the epistemology of cognizing true statements on self-consciousness,[5] or
(2.3) modelling the epistemology of cognizing true statements on the imagination.[6]

The other four categories I will call *concessive* negative or skeptical solutions, since they involve conceding both of the preliminary assumptions I and II, and then rejecting at least one of the other steps leading to the unacceptable conclusion.

2. Concessive Negative or Skeptical Solutions

(3) *Reject the classical necessity or apriority of mathematical truth.*

This entails accepting either

(3.1) the contingency of mathematical truth,or
(3.2) the aposteriority of mathematical truth.

[5]In (Hanna, 2006a Chapters 6 and 7), I work out Kant's idea that mathematical knowledge is grounded on reflective self-consciousness together with the imagination.

[6]One way of doing this would be via "plenitudinous platonism": For every consistently imaginable mathematical statement, there is a corresponding mathematical object. (See, e.g., Balaguer, 1998.) This construes imaginability as conceivability. But there are other ways of thinking about the imagination, e.g., Kant's conception of the productive imagination as a "schematizing" (i.e., mental modeling) capacity (Kant 1997, A84-147/B116-187, and esp. A120 n.). In (Hanna, 2006b, Chapter 6), I extend BD to logical knowledge, and then develop a strategy for solving the extended BD that starts with the thesis that a reasonable epistemology should be modeled on the imagination, not on perception. So by the classification scheme described here, strictly speaking, that earlier solution counts as a pre-emptive negative or skeptical solution. But to the extent that the present solution postulates the innate presence of mental modeling abilities in sense perception, it also postulates the innate presence of the capacity for *imagination* within the capacity for sense perception. So in that sense, the present positive or anti-skeptical solution is really only an extension and refinement of the earlier solution.

(4) *Reject the truth-making ontology of abstract mathematical objects.* (See, e.g., Shapiro 2000, Chapters. 6, 7, and 9.)

This in turn entails accepting either

(4.1) empirical or phenomenal idealism (whether communal or solipsist),
(4.2) intuitionism,
(4.3) formalism,
(4.4) conventionalism,
(4.5) fictionalism or some other form of nominalism, or
(4.6) non-cognitivist anti-realism.

(5) *Reject the thesis that sense perception involves causally efficacious, contact-involving or efficient, referentially direct, non-inferential, and spatiotemporal relations between human cognizers and the cognized objects.*

This in turn entails accepting either

(5.1) the replacement of causal efficacy by causal relevance,
(5.2) the counterfactual theory of causation,
(5.3) the probability-raising theory of causation,
(5.4) a non-causal theory of perception,
(5.5) an indirect causal theory of perception (whereby a perceptual subject *S* can sense perceive a universal *U* or type *T* just by standing in a direct causal sense perceptual relation to an instance of *U* or a token of *T*),
(5.6) referential descriptivism, or
(5.7) cognitive inferentialism.

(6) *Reject the thesis that abstract objects are causally isolated and inert.*

This in turn entails accepting either

(6.1) the causal relevance of abstract objects or
(6.2) the causal efficacy of abstract objects.

Obviously, some of these negative or skeptical solutions logically entail or logically exclude others. But at the same time many of the negative or skeptical solutions are also consistent with others, which gives rise to a large number of distinct possible *combined* negative or skeptical solutions.

This in turn makes the strategy of proving the rational superiority of a one or another of the given negative or skeptical solutions by attacking all the other possible negative or skeptical solutions somewhat strenuous, and possibly even unfeasible, given the usual limits on human time, energy, and patience.

In what follows in this paper, I will attempt to work out a *positive* or anti-skeptical solution to BD, but not explicitly to criticize or defeat the possible negative or skeptical solutions, which would require a separate book-length treatment on its own. As I said already, I call my solution to BD a "positive" or anti-skeptical one because it accepts Benacerraf's preliminary philosophical assumptions about the nature of semantics and knowledge, as well as all the basic premises of BD—captured in steps (1) to (6)—and then shows how we can, consistently with those very assumptions and premises, still reject the skeptical conclusion of BD—captured in step (7)—and also adequately explain mathematical knowledge. On the face of it, any positive or anti-skeptical solution should have a distinct rational edge over any negative or skeptical solution because only a positive or anti-skeptical solution will adequately preserve the rational force of all the original philosophical intuitions that generated the dilemma in the first place. If any of these intuitions did not have rational force, then BD would not be a *genuine* dilemma. So the fact that we *do* take BD seriously clearly entails that if there really *is* a positive or anti-skeptical solution, then prima facie *it* will trump any of the negative or skeptical solutions.

III. Benacerraf's Dilemma and Kantian Structuralism

> *Number* . . . is a representation that summarizes the successive addition of one homogenous unit to another. Number is therefore nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general, because I generate time itself in the apprehension of the intuition.
>
> –I. Kant (Kant 1997, A142–143/B182)

> Time provides a universal source of models for the numbers. . .. What would give time a special role in our concept of *number* which it does not have in general is not its necessity, since time is in some way necessary for all concepts, nor an explicit reference to time in numerical statements, which does not exist, but its sufficiency, because the temporal

order provides a representative of the number which is present to our consciousness if any is present at all.

–C. Parsons (Parsons 1983, 140)

The key to achieving a positive or anti-skeptical solution to BD, I think, is precisely how one interprets step (4) in my reconstruction, which says:

(4) Given (1) and (3), our standard, uniform semantics of natural language, as applied to true mathematical statements, commits us to a truth-making ontology of abstract mathematical objects and also to the non-empirical knowability of these statements.

It is *very* natural, and all-too-easy, to interpret the notion of "a truth-making ontology of abstract mathematical objects" in terms of classical Platonism. Classical Platonism about mathematics says that mathematical objects, which are the truth-makers of mathematical statements, have a mind-independent, substantial existence in a separate non-spatiotemporal realm, and that their nature is strictly determined by intrinsic non-relational properties of those objects. In short, classical Platonism interprets mathematical objects as what Kant would have called *things-in-themselves*. (See Hanna 2006a, esp. Chapters 1, 2, 3, 4, and 6.) This classical Platonist interpretation of the truth-making ontology of abstract mathematical objects postulated in step (4), I think, is precisely *the snake in the Garden of Eden*, by which I mean that I think that this interpretation is precisely the false and vitiating assumption which leads inevitably to Benacerraf's Dilemma and to skepticism, and I hereby reject it.

Granting that rejection as a starting point, my positive or anti-skeptical Kantian phenomenological solution to Benacerraf's Dilemma—as I previewed it in section I—then has two parts:

(1) *Kantian Structuralism*, which says

- (1.1) that the natural numbers are essentially positions or roles in the mathematical natural number structure provided by Peano arithmetic,
- (1.2) that the mathematical natural number structure provided by Peano arithmetic is abstract only in the sense that it is

transcendentally ideal, which is to say that this structure is identical to the formal structure of time insofar as we consciously represent it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human understanding of Peano arithmetic, and

(1.3) that in our actual world, the unique, intended model of the natural number structure provided by Peano arithmetic is just the set of manifestly real directly perceivable spatiotemporal material objects—the natural inhabitants of Parsons's "fuzzy *Lebenswelt* with its everyday objects"—insofar as *they are the role players of the Peano-arithmetic-specified natural number roles* in the abstract formal structure of time as we consciously represent it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human understanding of Peano arithmetic.

(2) *The Husserl-Wittgenstein Theory of Logical and Mathematical Self-Evidence* (the HW Theory), which holds that a priori knowledge in logic and mathematics is the joint product of two rational human abilities operating in tandem:

(2.1) a rational human agent's *mental-model-manipulating* abilities, which are innately specified in the agent's mind and also inherently present, as necessary ingredients, in ordinary sense perception, and which entail her conscious cognition of phenomenologically self-evident formal non-conceptual structures of human sense perception, together with

(2.2) that rational human agent's *logic-and-language-constructing* abilities, which are innately specified in the agent's mind and also inherently present, as necessary ingredients, in ordinary empirical conceptualizing and perceptual judgment, and which entail her conscious cognition of phenomenologically self-evident formal conceptual contents and specific patterns of logical inference in classical or non-classical logics.

In the rest of this section I want to unpack and rationally motivate the basic features of Kantian Structuralism. Then I will come back to the HW Theory in section IV.

Mathematical Structuralism, as an explanatory metaphysical thesis in the philosophy of mathematics—defended for example by Benacerraf himself, and in a different way by Stewart Shapiro (see, e.g., Benacerraf 1965; Shapiro 1997, 2000, Chapter 10), and most recently in another different way by Charles Parsons (Parsons 2008, esp. Chapters 3, 5, 6 and 9)—says that mathematical entities (e.g., numbers or sets) are not ontologically autonomous or substantially independent objects, but instead are, essentially, *positions* or *roles* in a mathematical structure, where a mathematical structure is a complete set of formal relations and operations that defines a mathematical system. What counts as an individual object of the system is thereby uniquely determined by the system as a whole—that is, any such individual object is identical to whatever possesses a specific set of intrinsic structural system-dependent properties. So every individual object of the system is essentially a role in the relevant mathematical system, and thus strongly metaphysically dependent on the whole system.

The significant philosophical payoffs of mathematical Structuralism are twofold. First, Structuralism gets between Platonism and Nominalism, because according to Structuralism mathematical objects are metaphysically absorbed into mathematical structures, hence they lack independent existence (contra Platonism), and yet it is also not true that there are no mathematical objects (contra Nominalism) since the objects continue to exist in a theoretically transformed way *as* roles in the structure. Second, because according to Structuralism the mathematical objects, as embedded in the relevant mathematical structure, continue to have whatever metaphysical status the relevant embedding structure has, then there is no longer any serious metaphysical "identity problem" of precisely *which* objects should be identified with the natural numbers, since we look to the embedding structures and not to the objects for any relevant metaphysical identity conditions.

In a way that is highly analogous to Functionalism in the philosophy of mind (see Block 1980b; Kim 2006, Chapters 5 and 6), there are at least two distinct ways we can interpret mathematical Structuralism. On the one hand, we can identify mathematical objects with *the roles* determined by the mathematical system as a whole. Or on the other hand, we can identify mathematical objects with *the role players* of the mathematical roles determined by the system as a whole. Which interpretation of mathematical Structuralism should we accept?

In the analogous case of Functionalism in the philosophy of mind, I think that there is good reason to take the Role-Player interpretation seriously because we think that it is intuitively plausible to identify a mind with whatever it is that actually does all the things that cognitive systems are supposed to do, and not merely to identify it with the set of causally relevant abstract patterns or rules that actual cognitive systems follow. If a mind were merely identical with a set of causal-functional roles, then it would be open to the classical inverted qualia argument, Searle's Chinese Room argument, and Block's Chinese Nation argument (a.k.a. "the absent qualia argument") (see ibid., and also Block 1980a; Searle 1984), not to mention the deeper worry that causal relevance does not entail causal efficacy (see, e.g., Jackson 1996), which yields the unhappy result that even *the representational mind* would be epiphenomenal if the Roles interpretation were true. Correspondingly, and to use an everyday non-philosophical analogy now, it seems intuitively right to say that a hockey player is a person who actually and in a causally efficacious way does all the things that hockey players are supposed to do, according to the rules of hockey—and obviously, a real hockey player is *not merely* the same as a set of causally relevant abstract rules that hockey players follow. So if we want minds to be *real causal players*, as it were, in physical nature, not to mention being *really capable of qualitative conscious experience* in addition to mental representation, then I think that we should defend a *dual* Roles interpretation *and* Role-Player interpretation of Functionalism, as opposed to a Roles interpretation alone or a Role-Player interpretation alone. We should say that for *some* rational purposes, the mind should be identified with functional roles, and also that for *other* rational purposes, the mind should be identified with the role-players of the roles.

By analogy, then, and for essentially the same basic reasons, I will adopt a *dual* Roles interpretation *and* Role-Player interpretation of mathematical Structuralism, as opposed to a Roles interpretation alone or a Role-Player interpretation alone. We want the natural numbers to be identified for many rational purposes with their roles in the mathematical structure of Peano arithmetic. But for other rational purposes we also want the unique, intended model of Peano arithmetic *to be consciously knowable according to a reasonable epistemology*, which is the direct analogue of the problem of qualitative conscious experience for the Roles interpretation of Functionalism. And we also want natural numbers and true statements about natural numbers *to be applicable to the actual spacetime world*, which is the direct analogue of the problem of epiphenomenalism for the Roles interpretation of Functionalism

So as I see it, mathematical Structuralism should hold that mathematical objects are essentially the same, for some rational purposes, as *the roles* in a given mathematical structure, and also essentially the same, for some other purposes, as *the role players* of the specific mathematical roles in a given mathematical structure, and *not* reducible either to those roles themselves or to the role-players themselves. The roles tell us precisely what will *count* as the unique intended model of that mathematical structure, but they neither *exhaust* the total nature of the mathematical objects nor do they *eliminate* the objects altogether. The mathematical objects are *strongly superveniently determined* by the structure as regards the precise roles they play, but they are also *something over and above* the structure as regards their role-player status. Different objects can play the same mathematical roles; the same objects can play different mathematical roles; and as a consequence, there is no intelligible worry whether the natural number 12 is the same as or different from the real number 12. This metaphysical dependency relation between mathematical structure and mathematical object in Structuralism thereby provides a precise analogue of *natural or nomological strong supervenience*, as opposed to *logical or reductive strong supervenience*, in the philosophy of mind.

Now BD clearly and distinctly shows us that we do not want the numbers to be the kinds of abstract entities that are also unknowable things in themselves and inapplicable to the actual spacetime world, lest we render mathematical truth and knowledge impossible. Or otherwise put, BD clearly shows us that the abstractness of the numbers must somehow correlate directly with what is consciously knowable according to a reasonable epistemology. This is possible, I think, if (and perhaps also only if) the abstractness of the numbers is *not* the abstractness of independent objects in a causally inert non-spatiotemporal realm, but instead just the abstractness of the roles *in a non-empirical or* a priori *consciously-accessible* *cognitive structure*. On this philosophical picture, the natural numbers are abstract because they are essentially roles in a *transcendentally ideal* structure.

In other words, I am proposing a broadly Kantian version of what Parsons calls "non-eliminative structuralism" (Parsons 2008, 100–116). More specifically, however, I think that the natural numbers are essentially the same, for *some* rational purposes, as roles in the abstract structure provided by Peano arithmetic, when this is interpreted as certain kind of non-empirical or a priori consciously-accessible cognitive structure, and also that the numbers are essentially the same, for *other* rational purposes, as the role players of the natural number roles in the real spacetime world, i.e., the natural numbers are just the set of manifestly real directly perceivable material objects intrinsically embedded in actual spacetime, insofar as they fall under the elementary or Peano arithmetic of the natural numbers. I will come back to this thesis again shortly.

Even if we have decided to adopt a dual Roles interpretation and Role-Players interpretation of structuralism, there are also several further basic distinctions between different kinds of Mathematical Structuralism that need to be made more explicit. The two main divisions are these:

(a) *Reductive* Structuralism vs. (b) *Non-Reductive* Structuralism,
(c) *In Rebus* Structuralism vs. (d) *Ante Rem* Structuralism.

Reductive Structuralism, as I am interpreting it, says that the objects of the mathematical system are either strictly identical with various elements and relations of the system or logically supervenient on the whole system and thus *nothing over and above* the whole system. By contrast, Non-Reductive Structuralism says that the objects of the system are strongly supervenient on the whole system but still *something over and above* the whole system, hence neither strictly identical to various elements and relations of the system nor logically supervenient on the whole system. In other words, the Reductive vs. Non-Reductive distinction applies to the *objects* of mathematical structural systems. Correspondingly, the Role-Players interpretation, on its own, entails Non-Reductive Structuralism, and the Roles interpretation, on its own, is consistent with both Non-Reductive Structuralism and Reductive Structuralism.

In Rebus Structuralism, as I am interpreting it, says that both the existence and specific character of the mathematical system are necessarily dependent on and determined by material things in the natural world, and that the systemic structures are not only literally proper parts of those material things but also ontologically *non-detachable* and epistemically *non-abstractible* from them. By contrast, *Ante Rem* Structuralism says that the existence and specific character of the system are neither necessarily dependent on nor determined by the existence of material things, and that the systematic structures are both ontologically *detachable* and also epistemically *abstractible* from those material things, even if they are also literally proper parts of them. In other words, the *In Rebus* vs. *Ante Rem* distinction applies not to the objects of mathematical structural systems, but instead to the *structural systems* themselves. For example, *In Rebus* Structuralism would be defended by a mathematical structuralist who is both a *reductive or scientific naturalist* and also an empiricist/nominalist, like Hartry Field (see e.g., Field 1980, 1989), whereas *Ante Rem* Structuralism would be defended by a mathematical structuralist who is both a *platonist* and also a rationalist/realist, like Shapiro.

Significantly, and perhaps because of the example set by Field, Shapiro identifies Reductive Structuralism with *In Rebus* Structuralism, and Parsons identifies both Reductive Structuralism and *In Rebus* Structuralism alike with what he calls "eliminative structuralism" (Parsons 2008, 80–100). But strictly speaking, at least in principle, one could consistently defend both *In Rebus* Structuralism and Non-Reductive (a.k.a. "non-eliminative") Structuralism. Consider, e.g., a specifically *Wittgensteinian* mathematical Structuralism (Wittgenstein 1983), in which numbers are identified with the entities that play the roles specified by mathematical linguistic practices, and not identified with the practice-specified roles, and in which those living mathematical linguistic practices *themselves*, conceived as rule-systems, are the enframing mathematical structural systems in which mathematical objects are embedded as the role-players of the roles in the structures. This Wittgensteinian Structuralism would be both *in rebus* and non-reductive. I myself am not going to defend such a Structuralism. But the very possibility of it does have a relevant bearing on the HW theory of mathematical apriori knowledge that I will defend in section IV, because I do think that mathematical *knowledge* is partially determined by living mathematical linguistic practices, even if mathematical *truth* is not so determined.

The brand of Structuralism I favor, Kantian Structuralism, is a non-reductive and *ante rem* version of mathematical Structuralism, doubly based on the abstract formal structures of space and time insofar as we consciously represent them in sense perception, together with formal concepts and the ramified abstract formal structures of classical logic and conservative extensions of it, insofar as rational human agents are capable of understanding those, that intends to take the necessity and apriority of mathematical truths at face value and then metaphysically explain those semantic features in terms of transcendentally ideal spatiotemporal structures, conceptual structures, and logical structures. By sharp contrast to Kantian Structuralism, however, Field's Structuralism is both reductive and *in rebus* because it says that numbers are nothing over and above their being positions in modal structures and that mathematical truth is reducible to fundamental physical facts about the physical world. And by another sharp contrast to Kantian Structuralism, Shapiro's Structuralism is both reductive and *ante rem* because it says that numbers are nothing over and above their being positions in non-modal structures and that mathematical truth is reducible to non-physical facts about non-spatiotemporal classically platonic structures.

But more precisely, and with respect to the elementary arithmetic of the natural numbers, i.e., Peano arithmetic, in particular, Kantian Structuralism says the following:

(1) that the natural numbers are essentially roles in the mathematical natural number structure provided by Peano arithmetic,
(2) that the mathematical natural number structure provided by Peano arithmetic is abstract only in the sense that it is *transcendentally ideal*, which is to say that this structure is identical to the formal structure of time insofar as we consciously represent it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human understanding of Peano arithmetic, and
(3) that in our actual world, the unique, intended model of the natural number structure provided by Peano arithmetic is just the set of

> manifestly real directly perceivable spatiotemporal material objects—the natural inhabitants of Parsons's "fuzzy *Lebenswelt* with its everyday objects"—insofar as *they are the role players of the Peano-arithmetic-specified natural number roles* in the abstract formal structure of time as we consciously represent it in sense perception, together with all the formal concepts and other logical constructions, including specific logical inference patterns such as mathematical induction, needed for an adequate rational human understanding of Peano arithmetic.

In this way, Kantian Structuralism adequately explains why something that is *abstract, ideal,* and *necessary* like the elementary arithmetic of the natural numbers, i.e., Peano arithmetic, can really and truly apply to the hurly-burly *concrete, manifestly real,* and *contingent* world of rational human animals and other natural things and processes, and thereby really and truly apply to all the manifestly real directly perceivable material spatiotemporal objects in our actual world. According to Kantian Structuralism, since the formal structure of time as we consciously represent it in sense perception is intrinsic to all manifestly real directly perceivable material spatiotemporal objects, and since the formal structure of time as we consciously represent it in sense perception together with anything isomorphic to the formal structure of time as we consciously represent it in sense perception, is the unique, intended model of Peano arithmetic, it follows as a matter of synthetic a priori necessity that Peano arithmetic applies to all manifestly real, directly perceivable, material spatiotemporal objects. The abstractness, ideality, and necessity of Peano arithmetic is captured by the *number roles* in the composite structure of time and Peano arithmetic and its conservative extensions, insofar as it can be understood by rational human agents. Correspondingly, the concreteness, reality, and contingency of the things and people to which arithmetic applies is captured by the *number role players* in the composite structure of humanly cognizable time and humanly cognizable Peano arithmetic and its conservative extensions. Thus consciously-representable time-structure is the metaphysical glue that ineluctably binds Peano arithmetic to our manifestly real natural world; or to re-use Parsons's apt phrase, consciously-represented time-structure is precisely what get[s] us across the divide between the fuzzy *Lebenswelt* with its everyday objects and the sharp, precise realm of the mathematical, in terms of which mathematical conceptions of the physical world are developed.

Otherwise put, Kantian Structuralism clearly solves the classical *application problem* for the philosophy of arithmetic. (See Potter 2000.)

So I am now in a position to solve BD by using Kantian Structuralism. I will begin by supposing that the two preliminary assumptions of BD are true. That obviously satisfies steps (1) and (2) of BD. Then I will further suppose that Kantian Structuralism is true, and that it adequately explains the apriority and necessity of mathematical truth. This satisfies step (3) of BD. This in turn allows me to re-interpret the truth-making ontology of abstract objects described in step (4) of BD as the transcendentally ideal abstract formal structure of time, and of anything isomorphic to time, insofar as we consciously represent it in sense perception, together with the transcendentally ideal abstract formal structure of any classical logical system rich enough to capture Peano arithmetic and conservative extensions of it, insofar as it can be understood by rational human agents. This dual abstract structure is itself of course causally isolated and inert, which satisfies step (6) of BD. But this dual abstract structure is also *intrinsically temporal*, and in our actual world it uniquely determines the unique intended model of the natural number structure, which then *just is* the directly perceivable manifestly real material world of spatiotemporal objects *insofar as* they are the role players of the Peano-arithmetic-specified natural number roles in the abstract structure of time. So the dual abstract structure consisting of the consciously-representable abstract formal structure of time together with Peano arithmetic and its rationally understandable conservative extensions is *causally relevant*, even though it is not *causally efficacious*. Therefore in our actual world the unique intended model of the natural number structure is identical to the world of causally efficacious manifestly real, directly perceivable material spatiotemporal objects, which obviously solves the application problem for Peano arithmetic, and mathematical knowledge is thereby possible on the assumption that a reasonable epistemology of cognizing true statements is modeled on a theory of sense perception which includes

causally efficacious, contact-involving or efficient, directly referential, non-inferential, and spatiotemporal relations between human linguistic knowers and the known objects themselves, understood by me to be some or another version of *direct or naïve perceptual realism* (see, e.g., Martin 2006; Haddock and F. McPherson 2008; Byrne and Logue 2009), which satisfies premise (5) of BD. Hence if Kantian Structuralism is true, then all of (1)–(6) are true, but the unacceptably skeptical conclusion of BD—step (7)—is avoided, and mathematical knowledge is still possible.

Considered for a moment apart from its ability to help us achieve a positive solution to BD, and also to solve the classical application problem for arithmetic, what other reasons could we have for defending Kantian Structuralism? I think that there are at least four other very good reasons.

First, Kantian Structuralism offers a clean-and-simple solution to another important problem pointed up by Benacerraf, which is that many different models satisfy the abstract structure of any logical system rich enough to express Peano arithmetic, so the second-order logic of Peano arithmetic underdetermines the natural numbers.[7] Otherwise put, Benacerraf's *other* problem is that there seems to be in principle no way of determining or identifying just *which* of the many distinct models that satisfy the logic of Peano arithmetic is *really* the natural numbers. This is what Parsons calls the "multiple reduction" problem (Parsons 2008, 48), and what others, following Frege, have called the "Caesar" problem or the "identification" problem. According to Kantian Structuralism, however, the abstract formal structure of the asymmetric successively synthesized series of moments (or simple events) in time insofar as we consciously represent it in sense perception is *the unique, intended model* of Peano arithmetic. On this picture, a "standard" model of Peano arithmetic is any possible world in which either time as we consciously represent it in sense perception exists, or else something isomorphic to the time-structure exists. (See, e.g., Parsons 2008, 272–293.)

But then the part of the model that satisfies a particular natural number-role in the abstract system of Peano arithmetic *just is* anything in our actual world that occurs in time as we consciously represent it in sense perception *insofar as* it intrinsically instantiates the thermodynamically asymmetric successive serial structure of time insofar as we consciously

[7] (See Benacerraf, 1965). This problem, in turn, is closely connected to Frege's "Caesar" problem. (See Frege, 1953, p. 68.)

represent it in sense perception, and thereby plays at least some of the Peano-arithmetic-specified natural number roles. The natural numbers themselves exist in non-actual possible worlds as *the Peano-arithmetic-specified and temporally-specified natural number roles*, and in our actual world as the unique intended model of Peano arithmetic, namely *the totality of real-world Peano-arithmetic-specified and temporally-specified natural number role-players.* Now the actual inhabitants of time insofar as we consciously represent it in sense perception are directly perceivable, manifestly real material spatiotemporal objects that contain spatiotemporal intrinsic structural properties. So in our actual world, the unique intended model of the natural number structure is identical to the totality of directly perceivable, manifestly real material spatiotemporal objects insofar as they are the role players of the Peano-arithmetic-specified natural number roles in the abstract formal structure of time insofar as we consciously represent it in sense perception.

Second, if Kantian Structuralism can offer a unified solution to BD *and* Benacerraf's other problem, then that seems to be another important point in its favor. For as Benacerraf himself has argued, BD and Benacerraf's other problem are essentially interdependent. So an adequate solution to BD must *also* solve Benacerraf's other problem (Benacerraf 1996).

Third, Kantian Structuralism crisply explains why classical Logicism failed, and why it seems that the arithmetic of the natural numbers is not reducible to second-order logic plus the Peano axioms alone. According to Kantian Structuralism, the elementary or Peano arithmetic of the natural numbers can be determined only by the ramified logical formal structure of Peano arithmetic and its conservative extensions insofar as it can be understood by rational human agents, together with any formal structure that is isomorphic to the structure of time insofar as we consciously represent it in sense perception. To be sure, contemporary neo-Logicists have shown that adding Hume's Principle (which says that the number of Fs = the number of Gs if and only if there are as many Fs as Gs) to second-order logic plus the Peano axioms logically entails the elementary arithmetic of the natural numbers. (See Wrigh, 1983; Hale 1987; Hale and Wright 2001.) But it seems to be intelligibly arguable that Hume's Principle is *not* an analytic truth precisely because it *presupposes* the formal structure of time insofar as we consciously represent it in sense perception, and also whatever is isomorphic to the formal structure of time insofar as we consciously represent it in sense perception. If so, then ironically enough the actual success of neo-Logicism is metaphysically best explained by *Kantian* Structuralism, and not by postulating the analyticity of *Hume's* Principle, as the neo-Logicists have done.

Fourth, if that is true, then Kantian Structuralism would also crisply explain why, contrary to both classical Logicism and neo-Logicism, mathematical truths seem *not* to be analytically necessary truths, but instead *synthetic* a priori *truths*. One good reason for thinking that mathematical truths are not true in every logically possible world, hence not analytic, is the clear and distinct conceivability and hence logical possibility, of either

(1) worlds with *nothing whatsoever* in them—which would of course entail the non-existence of numbers in those worlds, and thus the non-truth of many sentences of Peano arithmetic in those worlds (Parsons 1983, 131; Shapiro 1998, 604), or
(2) worlds with *non-standard arithmetics* of the natural numbers in them, e.g., a world in which "plus" is replaced by Kripke's "quus"—which would of course directly entail the non-truth of many sentences of Peano arithmetic in those worlds. (See Kripke 1982)

If mathematical truths are necessarily true but not analytically necessary, then according to Kantian Structuralism the explanation for this striking fact is that the truth and meaningfulness of mathematical propositions presuppose the abstract formal structure of time insofar as we consciously represent it in sense perception, which is not itself a purely logical or conceptual fact that attaches to every logically possible world. On the contrary, the presence either of the abstract formal structure of time insofar as we consciously represent it in sense perception, or of some other abstract structure isomorphic to the abstract formal structure of time insofar as we consciously represent it in sense perception, in a given possible world, is *a special metaphysical fact* that attaches to only a *restricted class* of logically possible worlds, i.e., all and only the logically possible worlds in which the very same spacetime structure, causal-dynamic structure, and mathematical structure as that of our actual world, also exist. This is also the special class of possible worlds in which *consciousness like ours* is really possible (see Hanna and Maiese, esp. Chapters 1, 2 and 6, 7 and 8).

On this view, possible worlds without denumerable objects in them are all time-structureless worlds, and all time-structureless worlds are possible worlds without denumerable objects in them. So if Kantian Structuralism is true, then the metaphysical explanation for *Modal Dualism*—which is the classical Kantian thesis that there are two essentially different kinds of necessary truth, namely

(1) analytic necessary truth, i.e., truth about the kind of necessity which flows from the nature of logic and concepts, which thereby includes logical truth and conceptual truth, and
(2) synthetic necessary truth, i.e., truth about the kind of necessity which flows from the nature of things in the world, which thereby includes mathematical truth (Hanna 2001, Chapters 3, 4 and 5)

—comes along for free.

Now if Kantian Structuralism is true, then it fully explains how the elementary arithmetic of the natural numbers, i.e., Peano arithmetic, is true. What about the rest of mathematics? The clean-and-simple answer provided by Kantian Structuralism is that all of the rest of mathematics, *including* its most abstruse and ontologically rich parts—e.g., iterative set theory—can be built up from Peano arithmetic and the abstract formal structure of time insofar as we consciously represent it in sense perception, together with all the formal concepts, classical logical constructions, and specific patterns of logical inference required by those other parts of mathematics, encoded in standard mathematical linguistic practices, insofar as mathematical language can be understood by rational human agents. Leopold Kronecker famously said that God made the integers and everything else was done by humans. (See, e.g., Struik 1967, 160.) Kantian Structuralism is even more radically anthropocentric. According to Kantian Structuralism, *the formal constitution of rational human nature* made the natural numbers, and *logico-conceptual construction by rational human agents, together with their innate capacity for linguistic understanding* did all the rest.

Now of course the Kantian structuralist still needs to explain more precisely *how* mathematical a priori knowledge is possible. And that is where the HW Theory comes in.

IV. The HW Theory

As we have seen, the Husserl-Wittgenstein Theory of Logical and Mathematical Phenomenological Self-Evidence holds that a priori knowledge in logic and mathematics is the joint product of two rational human abilities operating in tandem:

(1) a rational human agent's *mental-model-manipulating* abilities, which are innately specified in the agent's mind and also inherently present, as necessary ingredients, in ordinary sense perception, and which entail her conscious cognition of phenomenologically self-evident formal non-conceptual structures of human sense perception, together with
(2) that rational human agent's *logic-and-language-constructing* abilities, which are innately specified in the agent's mind and also inherently present, as necessary ingredients, in ordinary empirical conceptualizing and perceptual judgment, and which entail her conscious cognition of phenomenologically self-evident formal conceptual contents and specific patterns of logical inference in classical or non-classical logics.

And as its name clearly indicates, there are two historical provenances for the HW Theory: Husserl's specifically *phenomenological* approach to the epistemology of necessary truth in *Logical Investigations*, and Wittgenstein's specifically *linguistic* approach to the epistemology of necessary truth in the *Tractatus*. The historico-philosophical task of correctly interpreting each of these books is both highly strenuous and highly tricky, and, especially in the case of the *Tractatus*, currently quite controversial. In this context, I want to bracket those hard interpretive questions, and just state what I take to be the deep epistemological ideas lying behind Husserl's doctrine of "categorial intuition" and also behind Wittgenstein's doctrine that "language itself prevent[s] every logical mistake" by virtue of the fact that "we *cannot* think illogically."

For our purposes here, Husserl's deep epistemological idea is that the abstract formal structures characteristic of logic or mathematics are immediately represented in our inherently non-conceptual, pre-reflectively conscious awareness of the logico- syntactic and sortal-semantic structures of the meaningful sentences we use to frame true logical or mathematical judgments, and that the truth of those judgments is directly verified in direct perceptual experience of the manifestly real and intrinsically spatiotemporal natural world. This direct verification, in turn, is *phenomenological self-evidence*.

To understand the notion of phenomenological self-evidence properly, we need to sketch the basic concepts of Husserl's early phenomenology. Phenomenology, as Husserl understood it in 1900 in the first edition of the *Logical Investigations,* is an elaboration of "descriptive psychology" in Brentano's sense. More precisely, phenomenology is the first-person, introspective, non-reductive philosophical psychology of consciousness and intentionality, as opposed to the natural science of empirical psychology (*LI* 5, §7). As a specifically *philosophical* psychology, its basic claims, if true, are non-logically or synthetically necessarily true and a priori.

As Husserl points out in Investigation 5, consciousness (*Bewusstsein*) is a subject's capacity for "lived experience" or *Erlebnis*, i.e., phenomenal awareness, together with her capacity for *intentionality*. Intentionality, in turn, is essentially the same as what Kant would have called "directed experience" or *Erfahrung*. So more comprehensively, as I will put it, consciousness is *subjective experience*.

Now all subjective experience, insofar as it is "directed experience," or intentionality, is either dispositionally or occurrently directed towards *targets* of various kinds—objects (of all sorts), events (including intentional actions), and subjects (including oneself or others). Conversely, all "directed experience" or intentionality is either dispositionally or occurrently conscious in the sense of phenomenal awareness or "lived experience." In turn, every conscious intentional mental state *M* has four individually necessary and jointly individuating features:

(1) *M* is a mental *act* (*psychischer Akt*) with its own "immanent content" or "act-matter" and its own specific character (i.e., phenomenal character) (LI 5, §§11, 14, 20),
(2) *M*'s mental act falls under a specific intentional *act-type* or "act-quality," e.g., perceiving, imagining, remembering, asserting, doubting, etc. (LI 5, §20),
(3) *M*'s mental has an intentional *target*, which at the very least has ontic status or "being" (*Sein*) and perhaps also actual existence or "reality"

(*Wirklichkeit)*, although this target need not necessarily have reality—hence intentional targets can include fictional objects, impossible objects, abstract objects, ideal objects, etc. (LI 5, §§11, 17, 20), and
(4) *M* has an intentional *meaning content* or "semantic essence" (*bedeutungsmässige Wesen*), which presents its target in a certain specific way, where this meaning content is either *propositional* or *referential* (LI 5, §§21, 31–36)

It is crucial to note that this general phenomenological analysis holds *both* for the intentionality of judgment and belief, which presupposes pure formal logic and necessarily requires the existence of natural language and the intentional subject's linguistic competence, *and also* for the intentionality of perception and other modes of sensory cognition such as imagination and memory, which do not presuppose pure formal logic or necessarily require the existence of natural language or linguistic competence.

In Investigation 6, Husserl argues that truth (*Wahrheit*) is the structural and semantic conformity of a judgment to the very fact that satisfies its propositional content, and that authentic knowing (*Erkennen*) or "self-evidence" (*Evidenz*)—whether authentic a priori knowledge or authentic a posteriori knowledge—is the sufficiently justified conscious intentional recognition of truth (LI 6, §§6–12, 20, 28, 36–39). Moreover, self-evidence has its own characteristic phenomenology. The essential structure of the phenomenology of self-evidence is the advance from "empty" intentions to "filled" intentions, where

(1) empty intentions are logico-linguistically structured propositional contents insofar as they are *conceptually understood* by an intentional subject to specify the very facts that *could or would* satisfy those contents and thereby *make* those propositions true, and
(2) filled intentions are logico-linguistically structured propositional contents insofar as the very facts that could or would satisfy them are also *non-conceptually intuited* by an intentional subject as *actually satisfying* those contents and thereby *making* those propositions true.

In other words, and now formulated in an explicitly Kantian way, for early Husserl the phenomenological profile of authentic knowledge or self-evidence is a systematic advance from conceptual "understanding" (*Verstand*) to non-conceptual "intuition" (*Anschauung*), and this holds whether the authentic knowledge is a priori or a posteriori, and whether the truth-making fact that is intuitively experienced in intentional fulfillment as satisfying the relevant propositional content is a non-empirical or ideal (necessary or possible) abstract fact, or an empirical or real (contingent) concrete fact.

In the case of non-empirical or ideal facts, then the non-conceptual intuition by which the fact is self-evidently known is a *categorial* intuition (*LI* 6, §§40–58). Categorial intuitions are intentional states containing phenomenal characters that specifically pick out the formal and structural elements of the very facts that are known via intentional fulfillment, either by means of formal elements of perceptual consciousness, or by means of formal elements of logico-linguistic consciousness. The two paradigmatic examples of this special sort of a priori intuition would be the way in which aggregates of directly perceived objects (say, beer bottles) are non-conceptually and pre-reflectively "subitized" into finite groups (say, groups of 5 or 7), and the way in which a state-of-affairs as described by a statement or judgment (say, "The twelve beer bottles are all lined up in the shelf on the wall") appears to have the very same grammatical form as the sentence used to describe it.

What this all means, again for our purposes here, is that when we use very simple arithmetic sentences like "7 + 5 = 12" in making statements like "7 + 5 = 12," we are non-conceptually and pre-reflectively consciously aware of a temporal flow of mental images associated with our visual or auditory cognition of those inscriptions or utterances. Indeed, recent empirical research on memory strongly indicates that the non-conceptual, pre-reflectively conscious phenomenal look and sound of language is processed separately from the propositional cognition of linguistic meaning (see Schacter 1990). For example, I can vividly recognize and remember the look or sound of German sentences and words—*Die Welt is alles, was der Fall ist* or *Wovon man nicht sprechen kann, darüber muss man schweigen* (as, perhaps, screeched by the brilliant Finnish absurdist composer and singer M.A. Numminen[8])—without recognizing or remembering what they mean. Thus the mathematical propositions that

[8] See (and hear) (Numminen, 2009).

we express by means of the *self*-conscious intentional *conceptual* acts of cognizing the linguistic meanings of arithmetic sentences is directly combined with a *non*-conceptual, *pre-reflectively* conscious grasp of the formal structure of experiential or lived time.

And in turn, whenever we directly perceive a configuration of manifestly real material objects in the natural world that partially confirms the arithmetic propositions we express—say, we see seven bottles of beer on the wall sitting alongside five more bottles of beer on the wall, yielding the look of twelve bottles of beer on the wall—the non-conceptual, pre-reflectively conscious direct sense perceptions of those manifestly real material objects and also the self-conscious epistemic perceptions based on those direct perceptions, together with their perceptual, imaginational, and memory-based synthesis in time as we explicitly or implicitly count them up, immediately delivers to us a phenomenological formal structure that is also isomorphic to the addition operation over the natural numbers 7 and 5 in the system of Peano arithmetic. That non-conceptual, pre-reflectively conscious visual experience is a categorial intuition in Husserl's sense that necessarily impresses itself upon us as *mathematically self-evident*, where "self-evident" also means "inherently compelling," and as thereby conferring a *defeasible* epistemic certainty (Giaquinto 2007): As a rational human conscious intentional subject, you cannot help believing the propositional content associated with precisely that non-conceptual, pre-reflectively conscious subjective visual experience, or categorial intuition, precisely because it is inherently compelling. But correspondingly, the statement "$7 + 5 = 12$" is true if and only if there really is an appropriate mathematical truth-maker in the actual world that makes it true. This Husserlian doctrine, I think, provides a robustly realistic phenomenological interpretation of the classical Cartesian idea of "clear and distinct intuition."

Correspondingly, as I see it, the Tractarian Wittgenstein's equally deep epistemological idea is that to have logical or mathematical a priori knowledge is just

(a) to be a conscious rational human agent who possesses an innate conceptual cognitive capacity for non-conceptually and pre-reflectively consciously constructing, understanding, and using natural languages:

> Human beings possess the capacity of constructing languages, in which every sense can be expressed, without having an idea of how and what each word means—just as one speaks without knowing how the single sounds are produced. Ordinary language is a part of the human organism and is not less complicated than it. (Wittgenstein 1981, prop 4.002, 61–63. Translation slightly modified) and

(b) then actually applying the meaningful logical and mathematical sentences of those natural languages—e.g., "7 + 5 = 12"—according to the implicit normative rules of logic and natural languages, to a world of directly perceivable manifestly real material objects whose configurations inherently satisfy those sentences.

So if, plausibly, we take early Wittgenstein's remarks about cognizing language to be anticipations of a broadly *Chomskyan* theory of language (e.g., Chomsky 1986), then non-conceptually, non-self-consciously, and thus "tacitly" consciously knowing the logical and mathematical parts of natural languages is just a sub-species of non-conceptually, non-self-consciously, and thus "tacitly" consciously knowing a natural language more generally. This is a priori knowledge in the mode of *knowing exactly but also only non-conceptually and pre-reflectively consciously how to construct and use the language according to categorically normative rules of human rationality* (Hanna 2006c, esp. Chapters 4, 5, 6 and 7), and not a priori knowledge in the mode of self-consciously knowing exactly *what* one is doing or *that* one is doing it, whenever one actually does it. Or in other words, Wittgenstein is adumbrating the notion of a *conceptually-driven but also non-conceptually and pre-reflectively conscious* a priori *logical and mathematical linguistic competence*.

According to the HW Theory then, our knowing mathematical truths by means of mathematical judgments involves the very same sorts of non-conceptual, pre-reflectively conscious but also conceptually-driven cognitive activities as knowing factual truths by means of ordinary linguistic perceptual judgments, in accordance with direct or naïve perceptual realism. In this way, our innate conceptual capacity for constructing, understanding, and using the logical and mathematical parts of natural language, together with our innate non-conceptual capacity for direct sense perception and pre-reflective consciousness, when conjointly triggered appropriately by the world of directly perceivable manifestly real material spatiotemporal objects, and when correctly conjointly implemented by us, just *is* mathematical a priori knowledge in the classical sense. That is, and more briefly: You know some mathematical truths a priori when you are both non-conceptually and pre-reflectively consciously and *also* conceptually and self-consciously thinking or talking about mathematics correctly, and furthermore the manifestly real natural world also uniquely satisfies the mathematical statements generated in your language of thought or in your outer speech.

It is plausible to think, for reasons supplied by classical Constructivist theories of arithmetic, that the precise class of arithmetic statements that would be satisfied under phenomenologically self-evident mathematical a priori knowledge is primitive recursive arithmetic, or PRA, which is a fundamental fragment of elementary or Peano arithmetic containing the quantifier-free theory of the natural numbers and the primitive recursive functions. (See Skolem 1967; Troelstra and Dalen 1998, 120–126; Hanna 2006a, Sec 6.2.) More precisely, it is plausible to think that our directly perceivable and linguistic access to the unique intended model of Peano arithmetic will not permit us to verify all of Peano arithmetic with phenomenological self-evidence. Peano arithmetic is of course defined by the following five axioms:

(1) 0 is a number.
(2) The successor of any number is a number.
(3) No two numbers have the same successor.
(4) 0 is not the successor of any number.
(5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers,

together with the primitive recursive functions (basic calculations) over the natural numbers—the successor function, addition, multiplication, exponentiation, etc. But axiom (5) is not verifiable in an inherently *non-conceptual* way, and on the contrary requires the inherently *conceptual* ability to grasp quantifications over all the numbers. Nevertheless, given our grasp of all the arithmetic statements covered by the first *four* axioms, together with a grasp of the primitive recursive functions, and thus for PRA, there is no need whatsoever for a further theory of sufficient justification by epistemic reasons, nor for any sort of reply to skepticism. Therefore PRA is phenomenologically self-evident in the Husserlian and Wittgensteinian sense, precisely because the cognitive abilities required to grasp it are inherently non-conceptual and pre-reflectively conscious, and fall within the scope of categorial intuition.

Presumably there are also humanly-graspable, categorially-intuitable structural analogues of PRA in elementary geometry, elementary set theory, and elementary logic—e.g., *Euclidean* geometry, *basic* set theory (see, e.g., Potter 1990, Chapter 3), and *monadic* logic. If so, then Euclidean geometry, basic set theory, and monadic logic are all phenomenologically self-evident *too,* along with PRA.

It is crucial to note that a priori knowledge in mathematics and logic far exceeds the scope of phenomenological self-evidence and categorial intuition. Non-self-evident a priori mathematical and logical knowledge—e.g., a priori knowledge in non-Euclidean geometry and topology, Zermelo-Fraenkel set theory, and classical first-order polyadic logic—is inferential, conceptual, and of course also defeasible. But non-self-evident mathematical and logical a priori knowledge presupposes the phenomenologically self-evident and categorially intuitable parts of mathematics and logic, and constantly draws upon them as it carefully advances from the less defeasible, virtually uncontested, and more epistemically secure domains, towards the more defeasible, more contested, and less epistemically secure domains. This epistemic advance from the self-evident a priori to the non-self-evident a priori is beautifully symbolically mirrored in the situation of Adam and Eve as they leave Paradise at the end of *Paradise Lost*, with a hard-won awareness of what is and what is not really possible for creatures like us, in our rational *human* condition:

> They looking back, all the eastern side beheld
> Of Paradise, so late their happy seat,
> Waved over by that flaming brand, the gate
> With dreadful faces thronged a fiery arms.
> Some natural tears they dropped, but wiped them soon;
> The world was all before them, where to choose

Their place of rest, and Providence their guide.
They hand in hand with wandering steps and slow,
Through Eden took their solitary way.[9]

We can now see that the HW Theory is breathtakingly elegant. It also coheres perfectly with Kantian Structuralism and direct perceptual realism. For if Kantian Structuralism and direct perceptual realism are both true, then the HW Theory makes *perfect sense,* precisely because our actual world of directly perceivable manifestly real material spatiotemporal objects intrinsically carries with it the abstract formal structures of the system of Peano arithmetic and its conservative extensions, and thus directly perceptually presents the system of natural numbers, i.e., the intended model of Peano arithmetic, via the self-evidence of primitive recursive arithmetic or PRA, to any rational human conscious intentional subject who is also competent in the mathematical parts of her own natural language.

V. Conclusion: Benacerraf's Dilemma Again and "Recovered Paradise"

If Kantian Structuralism, direct perceptual realism, and the Husserl-Wittgenstein Theory of logical and mathematical self-evidence are all true, then both of Benacerraf's preliminary philosophical assumptions about a "standard, uniform" semantics of natural language and a "reasonable" epistemology of cognizing true statements are true, and the other four steps of Benacerraf's Dilemma are also true, but the unacceptably skeptical conclusion does *not* follow. Mathematical a priori knowledge in at least the classical, Kantian sense still *is* possible. Kantian Structuralism together with direct perceptual realism *also* together solve the classical application problem for mathematics; they solve Benacerraf's other problem about what the numbers could not be; they explains why classical Logicism failed; and they account for the synthetic necessity of mathematical truth. All of these very important individual theoretical virtues then seem to me to add up very naturally to one great big sufficient

[9] (Milton, 1953a, p. 487, book XII, lines 641–649).

reason for accepting Kantian Structuralism, direct perceptual realism, and the HW Theory as a single package.

Moreover, the conjunction of Kantian Structuralism, direct perceptual realism, and the HW Theory yields a phenomenologically-enriched *Kantian logico-linguistic constructivism* as a serious alternative to classical Platonism about mathematics on the one hand, and also to all the more or less skeptical recent and contemporary theories of mathematics—i.e., the full range of pre-emptive or concessive negative solutions to Benacerraf's Dilemma—on the other. This, in turn, suggests a fundamental Kantian, Husserlian, and Wittgensteinian insight into the nature of a priori knowledge. Given this phenomenologically-enriched Kantian logico-linguistic constructivism, what is required for mathematical knowledge is just a linguistically competent, healthy, developmentally normal, and relatively mature rational human conscious intentional subject, who can grasp both the non-conceptual content of perception and also the conceptual and propositional content of statements or judgment, who has also learned the basics of PRA, and who is thus primed and ready for speaking her own natural language, and for non-conceptually and pre-reflectively consciously but also conceptually and self-consciously intaking her manifestly real world through direct sense perception. And that is *all* that is required. Mathematics, just like *logic*—as I have argued elsewhere (see Hanna 2006b), is an *exact science* and yet also inherently a *human* or *moral* science. In this way, by equally rejecting *both* classical Platonism *and* post-Benacerrafian skepticism about mathematical truth and knowledge, we find

> Eden raised in the waste wilderness.

So let us go forth and multiply. And of course also add, subtract, divide, and correctly perform the other primitive recursive functions over the natural numbers too.[10]

[10] I am very grateful to the organizers (especially Mirja Hartimo, Leila Haaparanta, and Sara Heinämaa) of and also the participants in (especially Juliette Kennedy and William Tait), the Phenomenology and Mathematics conference at the University of Tampere, Finland in May 07, where I presented an earlier version of this paper, for all their help—critical, editorial, philosophical, and otherwise.

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CHAPTER VIII

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