A Way to Compare Spillovers Generated by a Public Project *

Anna Rubinchik

University of Colorado at Boulder
Department of Economics
256 UCB
Boulder, CO, 80309-0256
Tel. (303) 735-0220
Fax (303) 492-8960
e-mail: Anna.Rubinchik@Colorado.edu

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Abstract

This paper suggests a way to compare the magnitude of spillovers generated by a public project. The criterion is based on Lorenz curves of the benefits distribution, which has a variety of desirable characteristics. The more evenly distributed benefits profiles correspond to more "global" public projects. The more "concentrated" ones are more "local" according to this criterion. The conditions for comparison are formulated in a discrete setting, which is used in a wide range of applicaitons.

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1. Motivation

The notions of "concentrated" benefits, "local" and "global" public goods appear in numerous contexts in public economics and political economy literature.[1] The extent, to which a public project is of national interest, dictates the rules to be used for their consideration by a government including intergovernmental cost sharing arrangements. Nevertheless, to the best knowledge of the author, there is no general and convenient way to compare the spillovers generated by a public project. This paper provides one.

2. Construction

Consider a public project that generates a vector of benefits $b = (b_1, .., b_N) \in \mathbb{R}^N_+$, where $N > 0$ is the size of the population.[2] It is reasonable to require that the

[1] For example, the seminal work by Oates (1972) offers an insight as to how to allocate responsibilities within a hierarchical government. The degree of spillovers generated by a public project is a crucial factor that determines the allocation.

[2] In many applications the focus is on the distribution of benefits across localities, or regions. In this case, of course, one could think of N as the number of regions populated by residents with identical attitudes towards public goods. In this sense, more concentrated benefits can be thought of being more "local" in the geographic sense.

way, in which the degree of externality is measured, should not depend on the identity of the individuals who benefit from the project. Therefore, without loss of generality assume that the vector is ordered: $b_1 \leq b_2 \leq ... \leq b_N$. One can think of the benefits generated by a public good as representing individual willingness to pay for this good, i.e., demand for public good.

Note that the way the benefits are spread may depend both on technology of provision of the public good and on the distribution of tastes (or individual characteristics) across the population. For example, a poetry contest aired by a public radio station is "global" in this framework, if most of the recipients like it somewhat the same. On the other hand, it is "local", if only a small fraction of them understands the language, whereas the people who do not are indifferent to the mere fact that the contest is being transmitted. In this example the technology of provision allows for the public good to be "global", but the tastes may constrain the spread of the benefits. Clearly, it can be technology that restricts the spillovers. For example, a river dam affecting only residents, who live within a certain distance to the river, is usually considered to be a "local" public good.[3]

[3]Obviously, if a country has strongly empathetic citizens, any limitation imposed by technology of provision is irrelevant.

Conventional wisdom suggests that a project that generates benefits profile $b^1 = (1, 1, 1, .., 1)$ is a global public good, because everyone is affected in the same fashion. On the other extreme, a public good associated with the profile $b^0 = (0, .., 0, 1)$ is a local public good (or a private good), as it benefits only one person. How can one compare the intermediate cases, in which *some* of the people are *somewhat* affected by a public project? Let us use the Lorenz order to build such a comparison.

Let us associate with every project, b, a Lorenz curve,

$$\begin{aligned} L_b &: [0,1] \rightarrow [0,1] : \\ L_b\left(\frac{k}{N}\right) &\equiv \frac{\sum_{j=0}^{k} b_j}{\sum_{r=1}^{N} b_r}, k = 0, .., N. \end{aligned} \tag{2.1}$$

The function $L_b\left(\frac{k}{N}\right)$ measures the ratio of benefits received by k *lowest* beneficiaries to the total benefit.[4]

In order to define the Lorenz curve for the intermediate values, I will use linear interpolation between points $\left(\frac{k}{N}, L_b\left(\frac{k}{N}\right)\right), k = 0, .., N$. Construct a profile

[4]Note that for this definition the convention is that $b_0 \equiv 0$.

$\tau = (\tau_0, \tau_1, .., \tau_N)$ for a given vector $b \in \mathbb{R}^N_+$,

$$\tau_i \equiv \frac{b_i}{\frac{1}{N}\sum_{j=1}^N b_j}, \tag{2.2}$$

where $b_0 = 0$, as before.

Then the Lorenz curve evaluated at some point $u \in \left(\frac{k-1}{N}, \frac{k}{N}\right)$ can be represented as

$$L_b(u) = \sum_{i=1}^{k-1} \int_{(i-1)/N}^{i/N} \tau_i dt + \int_{(k-1)/N}^{u} \tau_k dt,\ \ k = 1, .., N \tag{2.3}$$

We shall refer to the profile τ as "cost sharing" profile due to its relationship to the Lindahl taxes.[5] It is easy to check that this representation $(2.1 - 2.3)$ corresponds to the measure introduced by Lorenz (1905).

Finally, we are ready to introduce the key definition.

Definition 2.1. *Let $\hat{b}, \tilde{b} \in \mathbb{R}^N_+$ be two vectors of benefits from public projects. Then project $\hat{b}$ is more global than project $\tilde{b}$ iff*

$$L_{\hat{b}} \geq L_{\tilde{b}} \tag{2.4}$$

[5] Indeed, let the cost of a public project generating benefits $b = (b_1, .., b_N)$ be C. Assume that a beneficiary $i \in \{1, .., N\}$ has to pay a (Lindahl) tax $\tau_i NC$ if the project is provided. A project is accepted only under a unanimous consent of the N people. Then all the efficient projects will be accepted, while all the inefficient projects will be rejected. See Myles (1995), p.272-273 for an additional interpretation of the tax.

It is easy to verify that for the example with two "extreme" projects, $b^1 = (1, 1, 1, .., 1)$ and $b^0 = (0, .., 0, 1)$, the comparison works as expected: $L_{b^1} \geq L_{b^0}$, so that the public good, generating the same benefits for everybody is, indeed, "more global" than the one generating a benefit to a single individual according to this order. In the next section I will demonstrate that this order corresponds to an intuitive notion of a public project being "more global" than another for intermediate cases in between the two extremes, b^1 and b^0.

A useful sufficient condition for two benefit vectors to be Lorenz ordered is based on the single crossing property of the corresponding cost sharing profiles. This condition is a counterpart of the single crossing for the generalized Lorenz order shown in Ramos et al. (2000). The proof of sufficiency is provided in lemma $A.1$ in the appendix.

Definition 2.2. *Profiles τ and τ' exhibit 'single crossing' property on $\{0, .., N\}$ if there exist integers $\underline{m}, \bar{m}$ such that $0 < \underline{m}, \bar{m} \leq N$; $\bar{m} - \underline{m} \leq 1$ and*

$$\bar{m} = \min \{m : \tau_m \leq \tau'_m\}; \quad (2.5)$$

$$\underline{m} = \max \{m : \tau_m \geq \tau'_m\}. \quad (2.6)$$

Example 2.1. *To demonstrate the definition of single crossing property, consider* *the cost sharing profiles that correspond to the benefits profiles* $\left(\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1\right)$ *and* $\left(\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\right)$

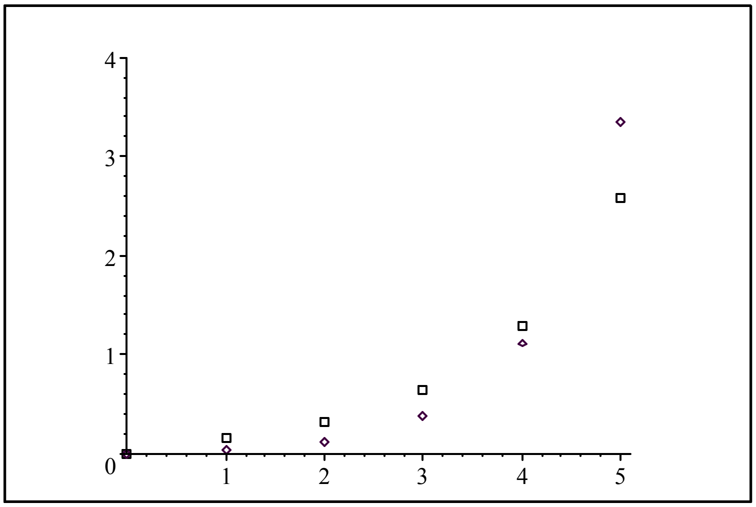

Figure 2.1: Profile τ *is depicted as "diamonds" and* τ' *is depicted as "squares". They cross once according to the above definition with* $\underline{m} = 4$, $\bar{m} = 5$.

3. Conditions for Comparison

The basis for the comparison suggested here is the shape of the benefits profile. If the impact of a project "spills over" more easily from the most affected people to the rest, the profile will be smoother. A local project generates a sharper distinction between the most affected people and the rest, so that primary beneficiaries are few, while the rest are almost unaffected. It is possible to represent the degree of spillovers in terms of the pace at which the differences in benefits grow along the benefits profile. More precisely, if for one project these differences grow more rapidly than for the other, the former is more local than the latter. Thus, it is natural to use the notion of convexification for the comparison.

In addition it will be shown that increasing benefits by the same amount for everybody, should make the profile of benefits "smoother", thus, making the spillovers more pronounced. This is in contrast to a proportional increase in benefits, which leaves the corresponding Lorenz curve unchanged, therefore having no effect on the project's ranking according to the definition 2.1.[6]

Finally, the last subsection contains necessary and sufficient conditions for the comparison. These conditions are well known in the income inequality literature.

3.1. Convexity of the Benefits Vector

Definition 3.1. *An ordered profile b is convex if $b_{k+1} - b_k \geq b_k - b_{k-1}$ for $k = 2, .., N$ under the convention that $b_0 = 0$.*

The profile is concave, if the inequality is reversed.

Example 3.1. *A profile $\tilde{b} = \left(\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1\right)$ is convex, while $\hat{b} = \left(\frac{2}{3}, \frac{8}{9}, \frac{26}{27}, \frac{80}{81}, 1\right)$*

[6] In the income inequality literature this corresponds to the notion of inequality as a relative concept, see Myles (1995) for an overview.

is concave.

Concave profile indicates that most of the beneficiaries are strongly affected, so it is associated with a global public good, convex profile is an indication of a local good: only a few individuals are benefitting from it, while all the rest are mostly indifferent.

One would expect that the suggested ordering will indicate that the concave projects are "more global" than the convex ones. In fact, a stronger result is true. For its demonstration we will need another notion.

Definition 3.2. *An ordered profile $\tilde{b}$ is a convex transformation of an ordered profile $\hat{b}$, if*

$$\frac{\tilde{d}_k}{\hat{d}_k} \geq \frac{\tilde{d}_{k-1}}{\hat{d}_{k-1}}, k = 1, .., N, \tag{3.1}$$

where

$$d_k = b_{k+1} - b_k,$$

provided $b_{k+1} - b_k > 0$ and $\tilde{d}_k = 0$ implies $\hat{d}_k = 0$.

Example 3.2. $\tilde{b} = \left(\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1\right)$ *is a (strict) convex transformation of* $\hat{b} = \left(\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\right)$.

As the example demonstrates, this transformation should produce a profile of benefits corresponding to a "more local" public project. This is the case in general, according to the following proposition.

Proposition 3.1. *If profile $\tilde{b} \in \mathbb{R}^N_+$ is a convex transformation of a profile $\hat{b} \in \mathbb{R}^N_+$, then*

$$L_{\hat{b}} \geq L_{\tilde{b}}. \tag{3.2}$$

Proof. First step is to show that if vector $\tilde{b}$ is a convex transformation of $\hat{b}$, then the corresponding cost sharing profiles, $\tilde{\tau}$ and $\hat{\tau}$, exhibit single crossing property, see definition 2.2. But by lemma A.1 in the appendix, this is sufficient for the desired inequality (3.2). Therefore, it is sufficient to prove the followong lemma. ■

Lemma 3.1. *Let $\hat{b}, \tilde{b} \in \mathbb{R}^N_+$ such that $\tilde{b}$ is a convex transformation of $\hat{b}$. Let $\hat{\tau}$ be the cost sharing profile generated by the benefit vector $\hat{b}$ and $\tilde{\tau}$ be the cost sharing profile of $\tilde{b}$. Then $\hat{\tau}$ and $\tilde{\tau}$ exhibit the single crossing property.*

Proof. First, note that $\tilde{\tau}$ and $\hat{\tau}$ have to 'cross' at least once. Indeed, they coincide at the 'initial point', $m = 0$, and sum up to unity:

$$\hat{\tau}_0 = \tilde{\tau}_0 = 0; \tag{3.3}$$

$$\sum_{i=1}^{N} \int_{(i-1)/N}^{N} \hat{\tau}_i = \sum_{i=1}^{N} \int_{(i-1)/N}^{N} \tilde{\tau}_i = 1. \tag{3.4}$$

If $\tilde{\tau}$ and $\hat{\tau}$ cross, it must be the case that for some $m < N$ either of the two conditions is satisfied,

$$\left\{\tilde{\tau}_{m+1} - \hat{\tau}_{m+1} \leq 0 \leq \tilde{\tau}_m - \hat{\tau}_m\right\}, \tag{3.5}$$

$$\left\{\hat{\tau}_{m+1} - \tilde{\tau}_{m+1} \leq 0 \leq \hat{\tau}_m - \tilde{\tau}_m\right\}. \tag{3.6}$$

Let us show that the intersection of the first type, (3.5), is not possible unless both inequalities hold as equality, i.e., $\tilde{\tau}_{m+1} = \hat{\tau}_{m+1}$ and $\tilde{\tau}_m = \hat{\tau}_m$. The proof will proceed under the assumption that both cost sharing profiles (and therefore, the original benefit profiles) are strictly increasing, so that $\tau_{k+1} - \tau_k > 0$ for any k. It is easy to reproduce the proof for weakly increasing profiles.

Assume that 3.5 holds for some $m > 0$. Then, if at least one of the inequalities is strict,

$$\frac{\tilde{\tau}_{m+1} - \tilde{\tau}_m}{\hat{\tau}_{m+1} - \hat{\tau}_m} < 1. \tag{3.7}$$

On the other hand, it is straightforward that if $\tilde{b}$ is a strict convex transformation of $\hat{b}$, then $\tilde{\tau}$ is a strict convex transformation of $\hat{\tau}$, i.e.,

$$\frac{\tilde{\tau}_k - \tilde{\tau}_{k-1}}{\hat{\tau}_k - \hat{\tau}_{k-1}} \leq \frac{\tilde{\tau}_{k+1} - \tilde{\tau}_k}{\hat{\tau}_{k+1} - \hat{\tau}_k}, k = 1, .., N, \tag{3.8}$$

so that the ratio

$$\frac{\tilde{\tau}_k - \tilde{\tau}_{k+1}}{\hat{\tau}_k - \hat{\tau}_{k+1}} \tag{3.9}$$

is increasing in the index k. Clearly, if all the inequalities (3.8) are satisfied as equalities, the profiles $\tilde{\tau}$ and $\hat{\tau}$ coincide (due to conditions $(3.3, 3.4)$), which is inconsistent with condition (3.7).

In the complementary case, from inequalities $(3.7, 3.8)$ it follows that

$$0 \leq \tilde{\tau}_m - \hat{\tau}_m < .. < \tilde{\tau}_1 - \hat{\tau}_1 < \tilde{\tau}_0 - \hat{\tau}_0, \tag{3.10}$$

which contradicts the fact that $\hat{\tau}_0 = \tilde{\tau}_0 = 0$ by construction.

For uniqueness it is left to show there can be only one intersection of the second type, (3.6). It is trivial that if condition (3.6) is satisfied for some m_0, (which requires $\tilde{\tau}_{m_0} \leq \hat{\tau}_{m_0}$) then it can not be true for $m_0 - 1$. Moreover it can not be satisfied for any $k < m_0$, as long as $\hat{\tau}_k \geq \tilde{\tau}_k$. If there is some index m' for which this inequality is reversed, i.e.,

$$\hat{\tau}_{m'} \leq \tilde{\tau}_{m'}, \tag{3.11}$$

then there must exist some $k' \geq m'$ such that

$$\hat{\tau}_{k'} \leq \tilde{\tau}_{k'} \text{ and } \hat{\tau}_{k'+1} \geq \tilde{\tau}_{k'+1}, \tag{3.12}$$

so that condition (3.5) is satisfied, indicating the first type of intersection, which is impossible by the previous argument. Thus, the intersection exists, it is unique and it is of type (3.6).

Let condition (3.6) hold for some $\underline{m} < N$. Let $\bar{m} = \underline{m} + 1$. Then, clearly,

$$\begin{aligned} \bar{m} - \underline{m} &\leq 1; \qquad (3.13) \\ \underline{m} &= \max\left\{m : \hat{\tau}_m \geq \tilde{\tau}_m\right\}; \\ \bar{m} &= \min\left\{m : \hat{\tau}_m \leq \tilde{\tau}_m\right\}, \end{aligned}$$

which corresponds to the single crossing property. ∎

3.2. Adding an Equal Amount to All the Beneficiaries

As discussed in the beginning of this section, increasing all the benefits by the same amount will make the benefits profile "smoother", thus should indicate an increase in the impact from the public project, making it more "global". The corollary formalizes this argument.

Corollary 3.1. *Let* $b \in \mathbb{R}_{+}^{N}$ *and* $b^{+} = (b_1 + a, b_2 + a, .., b_N + a)$, $a > 0$, *be benefit vectors. Then*

$$L_{b^{+}} \geq L_{b}. \tag{3.14}$$

Proof. Note that the cost sharing profile corresponding to b^{+} is

$$\tau_i^{+} = \frac{b_i + a}{\beta + a}, \tag{3.15}$$

where $\beta = \frac{1}{N}\sum_{i=1}^{N} b_i$.

It is easy to check that

$$\tau_i^{+} \geq \tau_i \Leftrightarrow \beta \geq b_i \tag{3.16}$$

Thus, let

$$\begin{aligned} \bar{m} &= \min\{m : \beta \leq b_m\}; \qquad (3.17) \\ \underline{m} &= \max\{m : \beta \geq b_m\}. \end{aligned}$$

Clearly, $\bar{m} - \underline{m} \leq 1$. Therefore, τ and τ^+ exhibit the single crossing by definition, τ^+ "crosses" τ from above. It follows that

$$L_{b^+} \geq L_b \tag{3.18}$$

by lemma $A.1$. ■

3.3. Spreading the Benefits

This subsection provides necessary and sufficient conditions for the comparison of Lorenz curves that has been known since the beginning of the XX^{th} century.[7] The results were derived for the measurement of income inequality. In the view of definition 2.1 and proposition 3.1 they can also be useful to compare the spread of the benefits generated by a public project.

Definition 3.3. *A square matrix P is said to be bistochastic if its entries are*

[7] Proposition 3.2 is by Hardy et al. (1929), Hardy et al. (1952). The equivalence of (i) and (iii) was proved by Muirhead (1903) in the case of nonnegative vectors $\hat{b}, \tilde{b}$. An additional proof of this result can be found in Berge (1963).

non-negative and each of its rows and columns sum up to one, i.e.,

$$P = \begin{bmatrix} p_{11}, .., p_{1N} \\ p_{21}, .., p_{2N} \\ ... \\ p_{N1}, .., p_{NN} \end{bmatrix}$$

such that $p_{ij} \geq 0$, *for all* i, j *and* $\sum_{i=1}^{N} p_{ij} = 1$ *for all* j; $\sum_{j=1}^{N} p_{ij} = 1$ *for all* i.

Proposition 3.2. *For ordered (column) vectors* $\hat{b}, \tilde{b} \in \mathbb{R}^N$ *the following conditions are equivalent:*

(i) $L_{\hat{b}} \geq L_{\tilde{b}}$;

(ii) $\hat{b} = P\tilde{b}$ *for some bistochastic matrix* P;

(iii) $\hat{b}$ *can be derived from* $\tilde{b}$ *by successive applications of a finite number of* $T - transformations$, *where*

$$T(z) = (z_1, .., z_{i-1}, \gamma z_i + (1-\gamma) z_j, .., z_{j-1}, \gamma z_j + (1-\gamma) z_i, .., z_N) \tag{3.19}$$

for some $1 \leq i < j \leq N$ *and* $\gamma \in [0,1]$, *such that* $\gamma z_i + (1-\gamma) z_j \leq \gamma z_j + (1-\gamma) z_i$.

4. Conclusions

This paper suggests a way to compare the magnitude of externalities generated by a public project. The criterion is based on Lorenz curves of the benefits distribution, which has a variety of desirable characteristics. The more evenly distributed benefits profiles correspond to more "global" public projects. The more "concentrated" ones are more "local" according to this criterion.

The importance of developing such a criterion stems for the recent contributions in fiscal federalism literature (see Besley and Coate (2000), Lockwood (2002) and Wrede (2002) among others), where the nature of public goods (whether they are more local or more global) often times dictates the desirable architecture of a government. A stylized approach towards modeling spillovers in this literature is to assume that the impact of a project constructed in a region is (proportionally) the same for all the rest of the regions. This note offers a way to compare a wide range of shapes that benefit profile generated by a public project can take. Moreover, the comparison does not distinguish between the "technology based" and "tastes based" spillovers, thus, providing a unified framework for the analysis.

Its disadvantage is that it limits attention to the projects with positive externalities, which can be easily eliminated with an additional restriction. If one wishes to focus on the "strength" of externalities, it is legitimate to compare the benefit profiles composed of absolute values of individual benefits. These profiles, obviously, satisfy the non-negativity constraint.

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A. Auxiliary Statements

Lemma A.1. *Let* τ, τ'*be two cost sharing profiles corresponding to the profiles of benefits* b *and* b'. *Assume* τ, τ' *exhibit single crossing property, so that*

$$\begin{aligned} \bar{m} - \underline{m} &\leq 1; \text{ where} && \text{(A.1)} \\ \bar{m} &= \min\{m : \tau_m \leq \tau'_m\}; \\ \underline{m} &= \max\{m : \tau_m \geq \tau'_m\}. \end{aligned}$$

Then the Lorenz curves L, L' *corresponding to* b, b' *can be ordered.*

Proof. It implies that

$$k \leq \underline{m}, \ \tau_k \geq \tau'_k; \tag{A.2}$$

$$k \geq \bar{m}, \ \tau_k \leq \tau'_k. \tag{A.3}$$

It follows from $A.2$ and from the representation 2.3 that for $k \leq \underline{m}$, the Lorenz curve corresponding to b lies above that for b',

$$L\left(\frac{k}{N}\right) = \sum_{i=k}^{N} \int_{(i-1)/N}^{i/N} \tau_i dt \geq \sum_{i=k}^{N} \int_{(i-1)/N}^{i/N} \tau'_i dt = L'\left(\frac{k}{N}\right). \tag{A.4}$$

Moreover, due to the condition $(A.3)$, for indexes higher than $\bar{m}$ profile τ_k lies below τ'_k, but they have both have to integrate to unity, $L(1) = L'(1) = 1$, it must be the case that the inequality $A.4$ holds for the rest of the indices,[8] $k \geq \bar{m}$,

$$L\left(\frac{k}{N}\right) \geq L'\left(\frac{k}{N}\right). \tag{A.5}$$

Trivially the inequality extends to the intermediate values of the argument, in the view of definition 2.3. ■

[8] One may want to consult theorem 6.4 in Arnold (1987) for a more detailed argument that supports the last step $A.5$.