Coproducts and colimits of κ-quantales

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BLAST

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We will extend the second point to κ-quantales, substituting the monoid multiplication for the meet operation. And we will use the first point on suitable extensions to give information about quotients of κ-quantales.

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- The ideals of a commutative ring with unit form a quantale.

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The distributivity $a \cdot \bigvee b_i = \bigvee(a \cdot b_i)$ in L can be interpreted as saying that the mappings $(x \mapsto a \cdot x) : L \to L$ preserve all suprema, and hence they are left Galois adjoints.

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The distributivity $a \cdot \bigvee b_i = \bigvee(a \cdot b_i)$ in L can be interpreted as saying that the mappings $(x \mapsto a \cdot x) : L \to L$ preserve all suprema, and hence they are left Galois adjoints. This gives rise to an operation $\to$ on L such that

$$ab \leq c \quad \text{iff} \quad a \leq b \to c.$$

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Proof.

Suppose aRb. Then

$$ac \leq x \to s \iff acx \leq s \iff bcx \leq s \iff bc \leq x \to s$$

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If aRb then $\mu_R(a) = \mu_R(b)$, and for every κ-morphism $h : L \to M$ such that $aRb \;\Rightarrow\; h(a) = h(b)$ there is a unique quantale morphism $\overline{h} : L/R \to M$ such that $\overline{h}\mu_R = h$. Moreover, $\overline{h}(a) = h(a)$ for all $a \in L/R$.

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2. *If U and V are pre-ideals then $U \cdot V = \{uv : u \in U,\ v \in V\}$ is a pre-ideal. This operation is associative and commutative. If the monoid is idempotent, i.e., a meet semilattice, then $U \cdot U = U$.*
3. $U \cdot S = U$.
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Define the mapping

$$\rho^0_{\kappa S} : S \to \mathfrak{F}^0_\kappa S \equiv (a \longmapsto [a], \ a \in S) .$$

Abbreviate $\rho^0_{\infty S}$ to ρ^0_S.

The free κ-quantale over a commutative monoid

Theorem

$\rho^0_{\kappa S} : S \longrightarrow \mathfrak{F}^0_\kappa S$ is the free κ-quantale over the commutative monoid S. That is, for every κ-quantale L and monoid homomorphism $h : S \longrightarrow L$ there is precisely one κ-morphism $f : \mathfrak{F}^0_\kappa S \longrightarrow L$ such that the diagram

$$\begin{tikzcd} \mathfrak{F}^0_\kappa S \arrow[r, "f"] & L \\ S \arrow[u, "\rho^0_\kappa S"] \arrow[ur, "h"'] & \end{tikzcd}$$

commutes.

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and let $\rho^\lambda_{\kappa L}(a) : L \longrightarrow \mathfrak{F}^\lambda_\kappa L \equiv (a \longmapsto \downarrow a,\ a \in L)$. Abbreviate $\rho^\lambda_{\infty L}$ to ρ^λ_L.

Theorem

Let L be a λ-quantale. Then $\mathfrak{F}_\kappa^\lambda L$ is a quantale with respect to the operations

$$U \cdot V = \downarrow \{uv : u \in U,\ v \in V\},$$

$$\bigvee_I V_i = \downarrow \left\{ \bigvee A : A \subseteq_\lambda \bigcup_I V_i \right\}$$

In fact, $\rho_{\kappa L}^\lambda(a) : L \longrightarrow \mathfrak{F}_L^\lambda$ is the free κ-quantale over L.

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Let L be a κ-quantale. An element $a \in L$ is called a λ-element if for all $A \subseteq_\kappa L$ such that $\bigvee A \geq a$ there is some $A_0 \subseteq_\lambda A$ such that $\bigvee A_0 \geq a$. The set of λ-elements of L is designated $\mathfrak{E}^\kappa_\lambda L$. This set is evidently closed under λ-joins,

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This result directly generalizes to κ-quantales Madden's corresponding result for κ-frames.

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Definition

A κ-ideal U on L is R-saturated iff

$$\forall\ a, b, c \in L\ \ (aRb \Longrightarrow (ac \in U \Longleftrightarrow bc \in U)).$$

We denote by $\langle A \rangle_R$ the smallest R-saturated κ-ideal containing a subset $A \subseteq L$.

κ-quantale quotients

Theorem

Let L be a κ-quantale, $\kappa > 0$, and let R be a binary relation on L. Then the R-saturated κ-ideals form a quantale in the order inherited from $\mathfrak{F}^\kappa L$, and the map

$$(a \longmapsto \langle a \rangle_R) : L \to \{\langle a \rangle_R : a \in L\}$$

*is a κ**Qnt**-quotient of L by the smallest κ-congruence containing R.*

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A pre-ideal $U \subseteq S$ is saturated iff it satisfies the following conditions.

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Let $[A]_R$ designate the smallest R-saturated pre-ideal containing a subset $A \subseteq S$.

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Let $\left[A\right]_R$ designate the smallest R-saturated pre-ideal containing a subset $A \subseteq S$. Let

$$\widetilde{L} \equiv \left\{\left[A\right]_R : A \subseteq_\kappa L\right\},$$

and let $\gamma_i : L_i \to \widetilde{L} \equiv \left(a \longmapsto \left[a\right]_R,\ a \in A\right)$.

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This directly generalizes to κ-quantales Johnstone's construction of the frame colimit.

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*Let $\kappa > 0$. A family $(\upsilon_i : L_i \to L)_J$ of κ-morphisms is a κ**Qnt** coproduct of the family $(L_i)_J$ iff it has these properties.*

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1. $\bigcup_J v_i [L_i]$ *generates L.*
2. *For any $I_0 \subseteq_\omega J$ and $I_1 \subseteq_\kappa J$, and for any choice of $a_i \in L_i$, $i \in I_0$, and $b_j \in L_j$, $j \in I_1$,*

$$\prod_{I_0} v_i (a_i) \leq \bigvee_{I_1} v_j (b_j) \Longrightarrow \exists\, i \in I_0 \cap I_1 \;\; (a_i \leq b_i) .$$

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Theorem

Let L be a κ-quantale, $\kappa > 0$, generated by a subset X. Then L is freely generated by X iff for any $X_0 \subseteq_\omega X$ and $Y \subseteq_\kappa X$, and for any choice of integers $n_x, m_y \in \mathbb{Z}^+$, $x \in X_0$, $y \in Y$,

$$\prod_{X_0} x^{n_x} \leq \bigvee_{Y} y^{m_y} \Longrightarrow \exists\, x \in X_0 \cap Y \ \ (n_x \geq m_y)\,.$$