TECHNISCHE UNIVERSITÄT DRESDEN

Relational Tame Congruence Theory

Mike Behrisch

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Outline

1. Preliminaries and notations
2. “Relational TCT” as a localisation theory
3. Examples of irreducible algebras / neighbourhoods

Outline

Polymorphisms and invariant relations

Let A be a set, $n \in \mathbb{N}$ and $m \in \mathbb{N}_+$.

$$\mathrm{O}_A^{(n)} := A^{A^n} \qquad \mathrm{R}_A^{(m)} := \mathcal{P}\left(A^m\right)$$

$$\mathrm{O}_A := \bigcup_{n\in\mathbb{N}} \mathrm{O}_A^{(n)} \qquad \mathrm{R}_A := \bigcup_{m\in\mathbb{N}_+} \mathrm{R}_A^{(m)}$$

For $f \in \mathrm{O}_A^{(n)}$ and $S \subseteq A^m$ (i.e. $S \in \mathrm{R}_A^{(m)}$):

$$f \rhd S \;:\Longleftrightarrow\; S \in \mathrm{Sub}\left(\langle A; f\rangle^m\right)$$

Polymorphisms and invariant relations

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For $f \in \mathrm{O}_A^{(n)}$ and $S \subseteq A^m$ (i.e. $S \in \mathrm{R}_A^{(m)}$):

$$\begin{aligned} f \rhd S \;&:\Longleftrightarrow\; S \in \mathrm{Sub}\left(\langle A; f\rangle^m\right) \\ &\Longleftrightarrow\; f \in \mathrm{Hom}\left(\langle A; S\rangle^n ; \langle A; S\rangle\right) \end{aligned}$$

Polymorphisms and invariant relations

For $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$:

$$\begin{aligned}
\operatorname{Inv}\langle A; F\rangle &:= \operatorname{Inv}_A F := \{\, S \in \mathrm{R}_A \mid \forall f \in F : \quad f \rhd S \} \\
&= \bigcup_{m \in \mathbb{N}_+} \operatorname{Sub}\left(\langle A; F\rangle^m\right) \\
\operatorname{Pol}\langle A; Q\rangle &:= \operatorname{Pol}_A Q := \{\, f \in \mathrm{O}_A \mid \forall S \in Q : \quad f \rhd S \} \\
&= \bigcup_{n \in \mathbb{N}} \operatorname{Hom}\left(\langle A; Q\rangle^n ; \langle A; Q\rangle\right) \\
\operatorname{Clo}(\mathbf{A}) &:= \operatorname{Pol}\operatorname{Inv}\mathbf{A} = T(\mathbf{A}) \quad \text{(term operations).}
\end{aligned}$$

Outline

Restricting algebras to neighbourhoods

A

Restricting algebras to neighbourhoods

$$\mathbf{A} \xrightarrow{\mathrm{Inv}_A} \underset{\sim}{\mathbf{A}} = \langle A; \mathrm{Inv}\,\mathbf{A} \rangle$$

Restricting algebras to neighbourhoods

$$
\begin{tikzcd}
\mathbf{A} \arrow[drr, "\mathrm{Inv}_A"] & & \\
\langle A; \mathrm{Clo}\,(\mathbf{A})\rangle & & \underset{\sim}{\mathbf{A}} = \langle A; \mathrm{Inv}\,\mathbf{A}\rangle \arrow[ll, "\mathrm{Pol}_A"]
\end{tikzcd}
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Restricting algebras to neighbourhoods

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\begin{tikzcd}
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& \underset{\sim}{\mathbf{A}}\restriction_U := \langle U; \underbrace{[\mathrm{Inv}\,\mathbf{A}]\restriction_U}_{=\left\{ S \cap U^{\mathrm{ar}\,S} \;\middle|\; S \in \mathrm{Inv}\,\mathbf{A}\right\}}\rangle
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\langle U; \mathrm{Pol}_U\,[\mathrm{Inv}\,\mathbf{A}]\restriction_U\rangle & \underset{\sim}{\mathbf{A}}\restriction_U := \langle U; \underbrace{[\mathrm{Inv}\,\mathbf{A}]\restriction_U}\rangle \arrow[l, "\mathrm{Pol}_U"]
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Definition (neighbourhood)

$U \in \mathrm{Neigh}\,\mathbf{A} \;:\Longleftrightarrow$ $U = e\,[A]$ for some $e \in \mathrm{Idem}\,\mathbf{A} := \left\{ g \in \mathrm{Clo}^{(1)}(\mathbf{A}) \,\middle|\, g^2 = g \right\}$

Restricting algebras to neighbourhoods

$$\begin{tikzcd}
\mathbf{A} \arrow[dr, "\mathrm{Inv}_A"] \arrow[dd, squiggly, bend right, "\big|_U"', "\equiv_{\mathrm{term}}"] & \\
\langle A; \mathrm{Clo}\,(\mathbf{A})\rangle \arrow[r, shift left, "\mathrm{Inv}_A"] & \underset{\sim}{\mathbf{A}} = \langle A; \mathrm{Inv}\,\mathbf{A}\rangle \arrow[l, shift left, "\mathrm{Pol}_A"] \arrow[d, two heads, "\upharpoonright_U"] \\
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![TU Dresden]

Restricting algebras to neighbourhoods

$$
\begin{tikzcd}[column sep=huge]
\mathbf{A} \arrow[dr, "\mathrm{Inv}_A"] \arrow[dd, squiggly, bend right=60, "\big|_U"', "\equiv_{\mathrm{term}}"] & \\
\langle A; \mathrm{Clo}\,(\mathbf{A})\rangle \arrow[r, shift left, "\mathrm{Inv}_A"] & \underset{\sim}{\mathbf{A}} = \langle A; \mathrm{Inv}\,\mathbf{A}\rangle \arrow[l, shift left, "\mathrm{Pol}_A"] \arrow[d, "\restriction_U"] \\
\langle U; \mathrm{Pol}_U\,[\mathrm{Inv}\,\mathbf{A}]\restriction_U\rangle \arrow[r, shift left, "\mathrm{Inv}_U"] \arrow[d, phantom, "\overset{!}{\Vert}"] & \underset{\sim}{\mathbf{A}}\restriction_U := \langle U; \underbrace{[\mathrm{Inv}\,\mathbf{A}]\restriction_U}_{=\left\{\, S\cap U^{\mathrm{ar}\,S} \;\middle|\; S\in \mathrm{Inv}\,\mathbf{A}\right\}}\rangle \arrow[l, shift left, "\mathrm{Pol}_U"] \\
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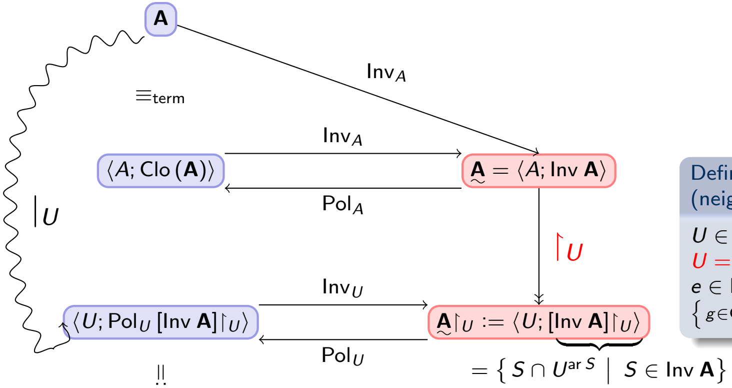
Restricting algebras to neighbourhoods



$$\mathbf{A}|_U = \left\langle U; \left\{ (e\circ f)\restriction^{U}_{U^{\mathrm{ar}\,f}} \;\middle|\; f \in \mathrm{Clo}\,(\mathbf{A})\right\}\right\rangle$$
$$= \left\langle U; \left\{ f\restriction^{U}_{U^{\mathrm{ar}\,f}} \;\middle|\; f \in \mathrm{Clo}\,(\mathbf{A}) \wedge f \rhd U\right\}\right\rangle$$

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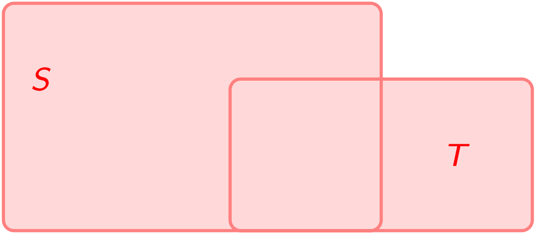
Distinguishing invariant relations



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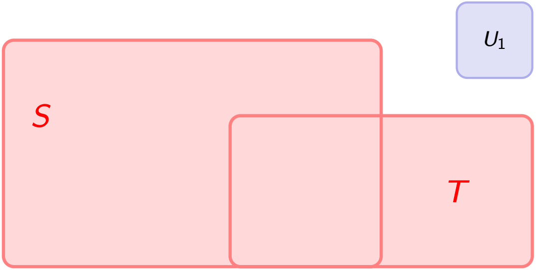

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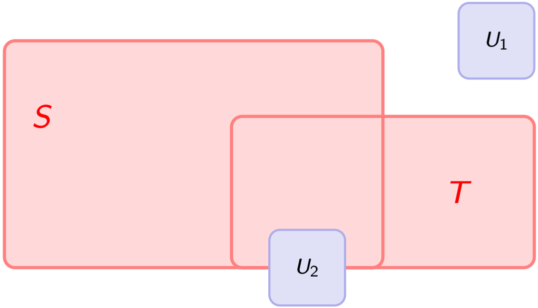

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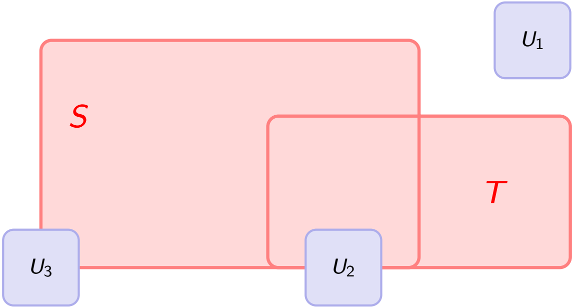

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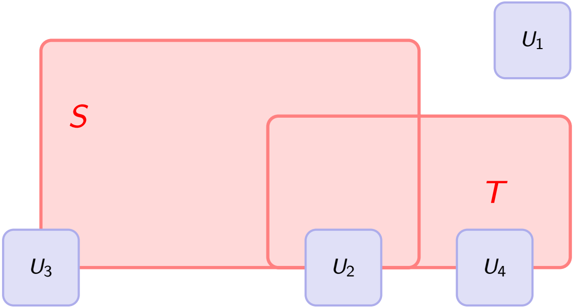

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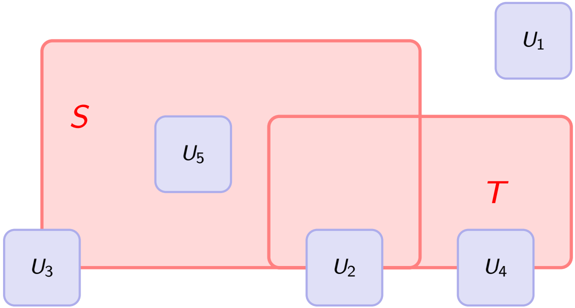

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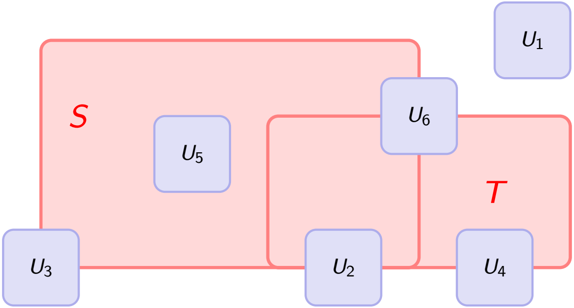

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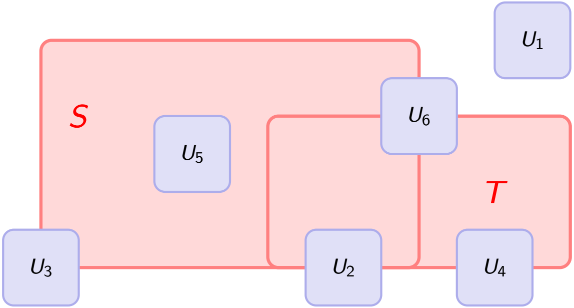

Definition (cover)

$\mathcal{U} \subseteq \mathrm{Neigh}\,\mathbf{A}$ cover of $\mathbf{A}$ iff for all $m \in \mathbb{N}_+$ and $S, T \in \mathrm{Inv}^{(m)}\mathbf{A}$ holds

$$S \neq T \quad \Longrightarrow \quad \exists U \in \mathcal{U}: \quad S{\restriction_U} \neq T{\restriction_U}.$$

Characterisation of covers

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Characterisation of covers

Theorem (Kearnes, A. Szendrei, 2001)

For an algebra $\mathbf{A}$ and a collection $\mathcal{U} \subseteq \operatorname{Neigh}\mathbf{A}$ of neighbourhoods of $\mathbf{A}$ (where each $U \in \mathcal{U}$ satisfies $U = e_U[A]$ for some fixed $e_U \in \operatorname{Idem}\mathbf{A}$) t.f.a.e.:

1. $\mathcal{U}$ is a cover of $\mathbf{A}$.

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1. $\mathcal{U}$ is a cover of $\mathbf{A}$.
2. There is some $q \in \mathbb{N}_+$, tuples $(U_1, \dots, U_q) \in \mathcal{U}^q$ and $(f_1, \dots, f_q) \in \left(\operatorname{Clo}^{(1)}(\mathbf{A})\right)^q$ and a term operation $\lambda \in \operatorname{Clo}^{(q)}(\mathbf{A})$ such that for all $x \in A$ holds

$$\lambda\left(e_{U_1} \circ f_1(x), \dots, e_{U_q} \circ f_q(x)\right) = x.$$

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3. There is some $q \in \mathbb{N}_+$ and a tuple $(U_1, \ldots, U_q) \in \mathcal{U}^q$ such that $\underset{\sim}{\mathbf{A}}$ is a retract of $\underset{\sim}{\mathbf{A}}\restriction_{U_1} \times \cdots \times \underset{\sim}{\mathbf{A}}\restriction_{U_q}$

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3. There is some $q \in \mathbb{N}_+$ and a tuple $(U_1, \ldots, U_q) \in \mathcal{U}^q$ such that $\underset{\sim}{\mathbf{A}}$ is a retract of $\underset{\sim}{\mathbf{A}}\restriction_{U_1} \times \cdots \times \underset{\sim}{\mathbf{A}}\restriction_{U_q}$, i.e. there are relational morphisms such that
$$\begin{tikzcd} \underset{\sim}{\mathbf{A}} \arrow[r, tail, "M"] \arrow[rr, bend right, "1_{\mathbf{A}}"'] & \prod_{i=1}^{q} \underset{\sim}{\mathbf{A}}\restriction_{U_i} \arrow[r, two heads, "\Lambda"] & \underset{\sim}{\mathbf{A}} \end{tikzcd}$$
commutes.

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When localisation is useless . . .

Definition (Irreducibility)

An algebra $\mathbf{A}$ is called irreducible, iff every cover $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ necessarily contains the neighbourhood $A \in \mathcal{U}$.

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$U \in \operatorname{Neigh} \mathbf{A}$ irreducible $:\Longleftrightarrow$ $\mathbf{A}|_U$ irreducible.

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Theorem (Kearnes, A. Szendrei, 2001)

A finite algebra $\mathbf{A}$ is irreducible iff the set of all unary non-bijective term operations is an invariant relation,
i.e. $\mathrm{Clo}^{(1)}(\mathbf{A}) \setminus \operatorname{Sym} A \in \operatorname{Sub}\left(\mathbf{A}^A\right)$.

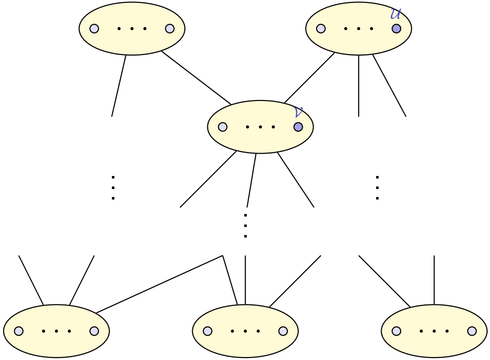
Refinement of covers

$\mathcal{V} \leq_{\mathrm{ref}} \mathcal{U}$

quasiorder

idea: smaller neighbour-hoods

Refinement of covers



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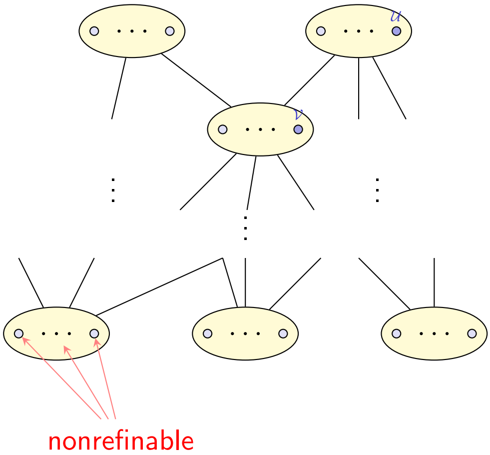

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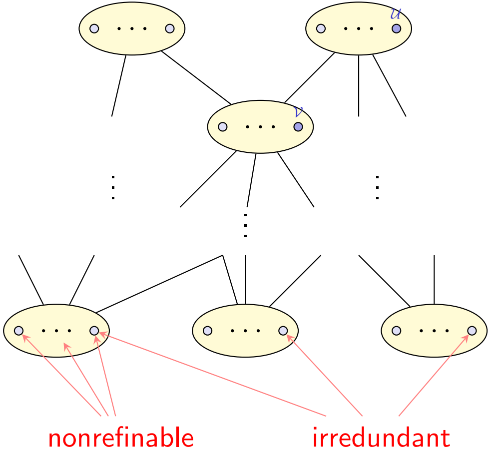

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Existence and uniqueness of covers

Theorem (Kearnes, A. Szendrei, 2001)

Every finite algebra $\mathbf{A}$ has got exactly one nonrefinable, irredundant cover $\mathcal{U}$ up to isomorphism. All neighbourhoods in $\mathcal{U}$ are irreducible.

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Consequence:

cover Decomposition of algebras in small parts (up to term equivalence)

uniqueness exactly one distinguished cover up to isomorphism consisting of irreducible neighbourhoods

irreducible algebras $=$ basic building blocks of finite algebras

check by $\mathrm{Clo}^{(1)}(\mathbf{A}) \setminus \mathrm{Sym}\, A \in \mathrm{Sub}\left(\mathbf{A}^A\right)$

Outline

Irreducible classes of algebras

i.e. all member are irreducible algebras

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7. vector spaces $\neq \{0\}$ over finite fields

Groups

Result

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Result (Waldhauser, 2009)

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$$\iff \underset{\sim}{\mathbf{A}} \rightsquigarrow \text{ type } 1 \,/\, 2$$

Irreducible subalgebra primal algebras

Let $\underset{\sim}{\mathbf{A}} = \langle A; Q \rangle$ be a unary relational structure where $Q \leq \langle \mathcal{P}(A), A, \cap \rangle$ as follows. $\implies$ $\underset{\sim}{\mathbf{A}}$ irreducible.

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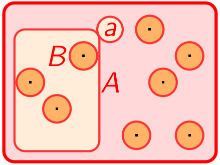

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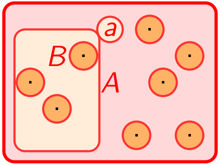

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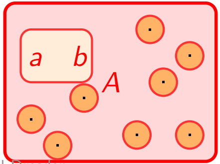

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6. Your ideas . . .

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