Boolean algebras over partially ordered sets

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Abstract

Being the crossroads between Algebra, Topology, Logic, Set Theory and the Theory of Order; the class of Boolean algebras over partially ordered sets were look at as one of the sources, providing over time, new insights in Boolean algebras. Some constructions and their interconnections will be discussed, motivating along the way a list of open problems.

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1 Introduction

A Boolean algebra is an algebraic structure (a collection of elements and operations on them obeying defining axioms) that captures essential properties of both set operations and logic operations.

Examples:

1. Power set algebras: $(\mathcal{P}(X), \cup, \cap, -, \emptyset, X)$
2. From logic: The Lendenbaum-Tarski algebra
3. From analysis: The algebra of projections
4. From topology: The regular open algebra

Stone representation for Boolean algebras

Every BA, $(A, +, -, 0, 1)$, is isomorphic to an algebra of sets: There is an algebra-embedding $s$ from $A$ into the power set algebra of $\text{Ultra}(A)$ given by:

$$s(a) = \{ U \in \text{Ultra}(A) : a \in U \}$$

where $\text{Ultra}(A)$ is the set of ultrafilters of $A$.

Countable Boolean algebras (Mostowski-Tarski (1939))

Any countable Boolean algebras is generated by a chain $(C, <)$ (i.e., a linear ordering).
2 Interval algebras

Let \((L, <)\) be a linear ordering. For \(t \in L\), set \(b_t := [t, \to)\). The subalgebra of \((\wp(L), \cup, \cap, -, \emptyset, L)\) generated by \(\langle b_t : t \in L \rangle\) is call the interval algebra over \(L\) and shall be denoted by \(\text{Int}(L)\). For technical reasons, we set \(L^\circ = L \cup \{\infty\}\) with \(\infty \notin L\).

1 Algebra

Each non zero element, \(x\), of \(\text{Int}(L)\) has a unique decomposition under each of the following form

\[ x = \bigcup_{i=0}^{n}[t_i, s_i] \quad \text{for} \quad t_i < s_i \quad \text{and} \quad s_i \in L^\circ \]  

\[ \mu^+(x) := |\text{supp}^+(x)| : = |\{t_i : i < n\}| \]  

\[ x = \bigtriangleup_{i=0}^{n}b_{t_i} \quad \text{for} \quad t_i \in L \]  

2 Topology

The Stone space of \(\text{Int}(L)\) is homomeorphic to the set of f initial segments \(I(L)\), of \((L, <)\), endowed with the topology inherited from \(2^L\).

\[ \phi : \text{Ultra}(\text{Int}(L)) \longrightarrow I(L) \]  

\[ \phi(U) = \{t : b_t \in U\} \]

3 (Posets/ lattices)

\((\text{Int}(L), <)\) is a distributive complemented lattice, where < the canonical boolean ordering on \(\text{Int}(L)\).

4 Combinatorics

1. Let the set of elements that are in \(x(\neq 0)\) be its positive support, \(\mu^+(x)\), in \(\text{Int}(L)\), when it is written in its normal form. Then For any natural number \(m \geq 3\) and \(x_1, \ldots, x_m\) in \(\text{Int}(L)\) pairwise disjoint we have:

\[ \mu^+(\sum_{i=1}^{m} x_i) = \sum_{1 \leq i < j \leq m} \mu^+(x_i + x_j) - (m - 2) \sum_{i=1}^{m} \mu^+(x_i) \]

2. (M. Rubin) If \(B\) is a subalgebra, of \(\text{Int}(L)\), of size \(\kappa\) uncountable regular, then \(B\) has a chain or an anti chain of size \(\kappa\).
5 Structure/generators

If \( \text{Int}(L) \) has a chain of size \( \kappa \) uncountable regular, then \( (L, \prec) \) has a chain, well ordered or anti-well ordered, of size \( \kappa \).

6 Special properties

A Boolean algebra \( B \) is superatomic if every nontrivial homomorphic image of \( B \) has an atom ( \( a \in B \) is an atom if \( 0 < a \) and there is no \( x \in B \) so that \( 0 < x < a \)).

The Cantor-Bendixon rank:

Let \( A \) be a Boolean algebra, denote by \( I(A) \) the ideal of \( A \) generated by the atoms of \( A \). For any ordinal \( \alpha \), we define a sequence \( I_\alpha \) of \( A \) as follows:

\[
A_\alpha = A/I_\alpha
\]

the \( \alpha^{th} \) derivative of \( A \), and let

\[
\pi_\alpha : A \rightarrow A_\alpha
\]

be canonical. Define \( I_0 = \{0\} \), \( I_{\alpha+1} = \pi_\alpha^{-1}[I(A_\alpha)] \), and for \( \lambda \) a limit ordinal, \( I_\lambda = \bigcup_{\alpha < \lambda} I_\alpha \). Finally, put \( I_\infty = \bigcup_\alpha I_\alpha \).

If \( A \) is not trivial, the first \( \alpha(A) \) so that \( A_{\alpha(A)} \) is finite, is called the Cantor-Bendixon rank of \( A \).

(Bonnet-Rubin-Sikaddour) Every superatomic (subalgebra of)interval algebra is generated by a well founded lattice.

Say that \( A \) satisfies the \( \text{qf} \)-property whenever every quotient of \( A \) is a factor i.e., if \( \pi : A \rightarrow B \) then there is \( Q \) so that \( A \simeq B \times Q \). Set \( X := \text{Ultra}(\text{Int}(L)) \), the Stone space of \( \text{Int}(L) \).

(Bekkali-Bonnet-Rubin) An interval algebra \( \text{Int}(L) \) satisfies the \( \text{qf} \)-property iff \( X := \text{Ultra}(\text{Int}(L)) \) is homeomorphic to \( \alpha + 1 + \sum_{i<n} \kappa_i + 1 + \lambda_i \), where \( \alpha \) is any ordinal, \( n < \omega \), for every \( i < \omega \), \( \kappa_i, \lambda_i \) are uncountable regular cardinals so that \( \kappa_i \geq \lambda_i \), and if \( n > 0 \) then \( \alpha \geq \text{Max}(\{\kappa_i : i < n\}) \).

4
7 Sub(Int(L))/Hereditary properties

For $A, B \in \text{Sub}(\text{Int}(L))$, define:

$$A \lor B := \text{the algebra generated by } (A \cup B), \quad A \land B := A \cap B.$$  

(S. Todorcevic, M. Rubin)

For any $B \subseteq \text{Sub}(\text{Int}(L))$, $\text{Sub}(B)$ is a complemented lattice.

(M. Rubin)

Every $B$ in $\text{Sub}(\text{Int}(L))$ is a retractive algebra i.e., if $\pi : B \rightarrow Q$ there is $r : Q \rightarrow B$ so that $\pi \circ r = \text{id}_Q$.

Recall that a pseudo-tree $(T, \prec)$ is a partially ordered set so that $\downarrow t := \{s \in T : s < t\}$ is a chain.

(J. Nikiel, S. Purisch, L. Heindorf)

Every subalgebra of an interval algebra is isomorphic to a pseudo-tree algebra $B(T)$, where $(T, \prec)$ is a pseudo-tree and $B(T) := \langle b_t := \{s \in T : s \geq t\}\rangle$.

(Bekkali)

Every pseudo-tree algebra $B(T)$ embeds in a very canonical way into an interval algebra.

8 Relation with other classes

The algebra finite-cofinite $FC(\aleph_1)$ is not isomorphic to an interval algebra.

9 Cardinal functions

See Cardinal Invariants on BAs by JD Monk for different types of cardinal functions on BAs and Cardinals functions in topology-ten years later by I. Juhász

**Problem 1:** Let, e.g., consider an interval algebra $\text{Int}(L)$. Is there a finite set of cardinal invariants that characterizes $\text{Int}(L)$ up to isomorphism?

10 Set Theory

**Problem 2:** Is it consistent to assume the existence of an uncountable interval algebra that is hereditary interval algebra?
2 Pseudo-treealgebras

Results in these sections are in collaboration with D. Zhani.

Recall that a pseudo-tree \((T, <)\) is a partially ordered set so that \(\downarrow t := \{s \in T : s < t\}\) is a chain.

Let \(b_t\) denotes the cone generated by \(t\) i.e., \(b_t = \{s \in T : t \leq s\}\). The subalgebra of \(\langle \wp(T), \cap, \cup, 0, 1, \prime \rangle\) generated by \(\langle b_t : t \in T \rangle\) is called the pseudo-tree algebra generated by \((T, <)\) and denoted by \(B(T)\). The case of \(B(T)\), where \(T\) is a well founded tree, were studied by G. Brenner.

1. Each non zero element of a pseudo-treealgebra \(B(T)\) has a unique decomposition using the set of generators of cones \(\langle b_t : t \in T \rangle\). Hence, one can define the positive support, \(\mu^+(x)\), of a non zero element \(x\), as in the case of interval algebra:

\[
\mu^+(\sum_{i=1}^{m} x_i) = \sum_{1 \leq i < j \leq m} \mu^+(x_i + x_j) - (m - 2) \sum_{i=1}^{m} \mu^+(x_i) \quad (\dag)
\]

**Problem 3:** Let \((Q, <)\) be a poset and \(B(Q)\) be the subalgebra of \(\wp(Q)\) generated by \(\langle b_t : t \in Q \rangle\). Assume there is \(l\) defined on \(B(Q)\) satisfying \((\dag)\). Is there a pseudo tree \((T, <)\) so that \(B(Q) \simeq B(T)\) and \(l = \mu^+\)?

2. Let \((T, <)\) be a pseudo-tree and set \(X = \text{Ultra}(B(T))\). Recall that the set, \(I_i(T)\), of initial chains in \((T, <)\) is a closed subspace of \(2^T\). Assuming, wlog, that \((T, <)\) has a single root, we have:

\[\text{Ultra}(B(T)) \simeq I_i(T)\]

3. We say that a pseudo tree \((T, <)\) is scattered whenever:
   • the order type of the chain of rational numbers does not embed in \((T, <)\);
   • The finite binary tree \(\leq_2\) does not embed in \((T, <)\) as a subtree.

A pseudo-treealgebra \(B(T)\) is superatomic iff \((T, <)\) is scattered
Pseudo-tree algebras is an example of a hereditarily class of boolean algebras. Looking closely one may refine this statement as follows:

Let \((T, <)\) be a pseudo-tree and set \(X = \text{Ultra}(B(T)) \simeq I_i(T)\). Denote by \(cl(Z)\) the topological closure of \(Z\) in \(X\). We say that \(b \in X\) is a bad point whenever there are three disjoint sets \(A, B, C\) such that:

- Either at least \(A\) is discrete and uncountable so that \(cl(A) = A \cup \{x\}\);  
- Or, \(cl(A) \cap cl(B) \cap cl(C) = \{x\}\), where \(A = \{a_\alpha : \alpha < \kappa\}\), \(B = \{b_\beta : \beta < \lambda\}\) are two linear orderings of uncountable cofinalities/coalinalities converging to \(x\) and \(C\) is an infinite sequence so that \(cl(C) = C \cup \{x\}\). Denote by \(\text{Bad}(X)\) the set of bad points in \(X\).

If \(B(T)\) is superatomic, then \(B(T)\) is isomorphic to an interval algebra iff \(\text{Bad}(X) = \emptyset\)

**Problem 4:** Is the previous statement true in more general setting?

**Problem 5:** (Faithfulness of pseudo-tree algebras)

Assume that there is a chain of type \(\theta\) in the pseudo-tree algebra \(B(T)\). Is there a pseudo-tree \(T^0\), so that \(B(T) \cong B(T^0)\), and \(T^0\) embeds either a chain of type \(\theta\) or \(\theta^*\).

**Problem 6:** (\(n\)-hard pseudo-tree algebras)

Let \(B\) be a boolean algebra and denote by \(r_{pt}(B)\) the least ordinal \(\beta\) so that \(B^{(\beta)} \simeq B(T)\) for some pseudo-tree \(T\); \(B^{(\beta)}\) being the \(\beta\)th Cantor derivative of \(B\).

Next denote by \(PT\) the class of all pseudo-tree algebras and set \(r_{pt}(B) := \delta(B, PT)\). For example, if \(B\) is a pseudo-tree algebra, we set \(\delta(B, PT) = 0 = r_{pt}(B)\), if \(B \notin PT\) and \(B^{(1)} \simeq B(T)\) for some pseudo tree \(T\); we set \(\delta(B, PT) = 1\), and we say that \(B\) is 1-hard pseudo-tree algebra.

Characterize, in the class of superatomic boolean algebras, \(n\)-hard pseudo-tree algebras for \(n < \omega\).
3 Free poset algebras

Let \( (P, \leq) \) be a poset. A free poset algebra over \( P \) is a BA \( A \) having \( P \) as a set of generators such that:

(1) \( p \leq q \) implies \( p \leq_A q \).

(2) For every BA \( B \) and every mapping \( f : P \to B \), if \( p \leq q \) implies that \( f(p) \leq f(q) \), then there is a homomorphism \( g : A \to B \) such that \( g \restriction P = f \).

Note that for \( P \) a poset consisting of isolated elements, that is with \( \leq \) the identity, a free poset algebra over \( P \) is just a free BA over \( P \). In particular, if \( P \) is infinite, then this algebra is atomless.

Proposition 1. Let \( P \) be a poset, and let \( C \) be a boolean algebra freely generated by \( P \). Let \( I \) be the ideal of \( C \) generated by \( \{x \cdot -y : x, y \in P, \text{ and } x \leq_P y\} \). Then \( C/I \) is a free poset algebra over \( P \).

Proposition 2. For any BA \( A \) the following conditions are equivalent:

(i) \( A \) is isomorphic to a free poset algebra.

(ii) \( A \) has a set \( G \) of generators such that \( 0, 1 \in G \), and if \( F; K \) are finite subsets of \( G \) and \( \prod_{x \in F} x \cdot \prod_{y \in K} -y = 0 \), then there exist \( x \in F \) and \( y \in K \) such that \( x \cdot -y = 0 \).

For any poset \( P \), a final segment of \( P \) is a subset \( M \) of \( P \) such that if \( p \in M \) and \( p \leq q \), then also \( q \in M \). For any poset \( P \), \( Fs(P) \) is the collection of all final segments of \( P \). Note that \( \emptyset \) is a final segment of \( P \).

Proposition 3. Suppose that \( P \) is a poset. Then \( Fs(P) \) is a closed subspace of \( P \). Moreover, \( \operatorname{clop}(Fs(P)) \) is a free poset algebra on \( P \).

Proposition 4. Every free poset algebra is a semigroup algebra.

Let \( P \) be a poset. An antichain in \( P \) is a collection of pairwise incomparable elements of \( P \). \( \operatorname{Ant}(P) \) is the set of all finite antichains of \( P \). Note that \( \emptyset \in \operatorname{Ant}(P) \), and \( \{p\} \in \operatorname{Ant}(P) \) for all \( p \in P \). We define a relation \( \preceq \) on \( \operatorname{Ant}(P) \) by:

\[ \sigma \preceq \tau \iff \forall p \in \sigma \exists q \in \tau (q \leq p) \]

Theorem 5. Let \( P \) be a poset. Then:

(i) \( (\operatorname{Ant}(P), \preceq) \) is an upper semi lattice.

(ii) For each \( p \in P \) let \( f(p) = \{p\} \). Then \( f \) is an order anti-isomorphism of \( P \) into \( \operatorname{Ant}(P) \).

(iii) \( B((\operatorname{Ant}(P), \preceq)) \) is isomorphic to a free poset algebra over \( P \).
4 Semi-group algebras

The following theorem gives a concrete construction of semi-group algebras. For let \((M, \wedge)\) be an idempotent semi-lattice with 0 and 1, and let \(A\) be the free Boolean algebra with free generators \(x_p\) for \(p \in M\), and let \(I\) be the ideal generated by the set \(\{(x_p \cdot x_q) \triangle x_{p \wedge q} : p, q \in M\}\).

Recall that a set \(H \subseteq B\) is a disjunctive set whenever:

i. \(0 \notin H\);

ii. For all \(h, h_1, \ldots, h_p \in H\) \(\left[ h \leq h_1 + \cdots + h_p \rightarrow \text{There is } i \text{ so that } h \leq h_i \right]\)

Recall that \((S, \cdot)\) is called a semi-lattice whenever "\(\cdot\)" is commutative, associative, and \(x^2 = x\) for all \(x \in S\).

**Theorem** The following statements are equivalent for any Boolean algebra \(B\).

i. \(B\) is isomorphic to an upper semi-lattice algebra.

ii. \(B\) is generated by \(H \subseteq B\) so that: \(0 \notin H, H\) is disjunctive, containing 1 and closed under multiplication.

**Theorem (D. Monk)** The following are equivalent.

i. \(A/I\) is a semi-group algebra;

ii. Every semi-group algebra is isomorphic to some \(A/I\) as above;

iii. The Stone space \(\text{Ult}(A/I)\) is homomorphic to \(F(S)\), where \((S, \wedge)\) is a unitary meet semi-lattice.

**Corollary** Every semi-group algebra is isomorphic to an upper semi-lattice algebra.

**Note:** If \((T, <)\) is a pseudo tree then \(H := \{b_t : t \in T\}\) generates \(B(T)\). Notice that \(0 \notin H, 1 \in H, H\) is disjunctive; but \(H\) is not closed under "\(\cdot\)". Hence one cannot draw the conclusion that \(B(T)\) is a semigroup algebra and therefore it is an upper semi-lattice by the previous theorem. Actually, there are examples where \(B(T)\) is isomorphic to an upper semi-lattice algebra.

**Problem 7:** Characterize atomless \(B(T), T\) is a pseudo tree, that is isomorphic to an upper semi-lattice algebra.

**Problem 7’:** \((n\text{-almost pseudo-treealgebras})\)

Denote by \(Pt\) the the class of pseudo-treealgebras. Call a tail algebra \(B(T)(= \langle b_t : t \in T \rangle)\) an \((n\text{-almost pseudo-treealgebra})\), whenever

\[ b_u, b_v \text{ is a finite set of size } n \text{ for all } u, v \text{ incomparable in } T. \]
• For $n$ finite or countable, characterize upper semi lattices $T$ so that $B(T)(=\langle b_t : t \in T \rangle)$ is an ($n$–almost pseudo-treealgebra) isomorphic to a pseudo-treealgebra.

• Are ($n$–almost pseudo-treealgebras) retractive algebras? for $n \in \omega$. 
5 Upper semi lattices algebras

Let \((T, \leq)\) be a partially ordered set and consider the subalgebra \(B(T)\) of the power set of \(T\), \(\mathcal{P}(T)\) generated by \(\{b_t : t \in T\}\), where \(b_t := \{x \in T : t \leq x\}\).

Recall that an upper semi-lattice poset \((T, \leq)\) is so that \(l.u.b.\{x, y\} := x \lor y\) exists in \((T, \leq)\) for all \(x, y \in T\).

Next theorem characterizes \(\text{Id}(T)\), for any upper semi-lattice \((T, \leq)\).

**Theorem** Let \(B\) be a Boolean algebra and set \(X = \text{Ult}(B)\) its Stone space. The following statements are equivalent.

i. \(B\) is isomorphic to \(B(T)\), where \((T, \leq)\) is an upper semi-lattice, with a least element.

ii. \(X\) is homeomorphic to \(\text{Id}(T)\), the set of ideals of an upper semi-lattice \(T\) with a least element, endowed with Tychonoff’s topology inherited from \(2^T\).

iii. \(X\) is homeomorphic to \(\mathcal{F}(S)\), the set of filters over \(S\), where \(S\) is a unitary semi-lattice, endowed with Tychonoff’s topology inherited from \(2^S\).

iv. There is a multiplication “\(\cdot\)” on \(X\) so that \((X, \cdot)\) is a unitary semi-lattice and “\(\cdot\)” is a continuous mapping on \(X \times X\) (i.e., \((x, y) \rightarrow x \cdot y\) is continuous).

**Theorem** For any poset \((T, \leq)\) the Stone space \(\text{Ult}(B(T))\) of \(B(T)\) is \(\overline{\text{Id}(T)}\) up to a homeomorphism.

5.1 Set of initial sections of a poset

Let \((P, \leq)\) be a poset, with a least element. Denote by \(\mathcal{I}(P)\) the set of non empty initial sections of \(P\). \(\mathcal{I}(P)\) is a Boolean space.

In this section, we state conditions under which \(\text{Id}(T)\), where \((T, \leq)\) is a poset, can be represented by \(\mathcal{I}(P)\), up to a homeomorphism, for some poset \(P\) and conversely.

Notice that, when \(P\) is a chain, we have \(\mathcal{I}(P) = \text{Id}(P)\).

**Definition**

Let \((T, \leq)\) be an upper semi-lattice. An element \(a \in T\) is prime whenever for all \(c, d \in T\) \((a \leq c \lor d \rightarrow a \leq c\) or \(a \leq d\)). \(\text{Prim}(T)\) shall denote the set of prime elements of \(T\).

Also, we say that \(\text{Prim}(T)\) is \(\lor\)-generating set for \(T\), whenever for every \(t \in T\)
there are \( t_1, \ldots, t_n \in \text{Prim}(T) \) so that \( t = t_1 \lor \cdots \lor t_n \).

**Theorem** For any poset \((P, \leq)\), with a least element, there exists an upper semi-lattice \( T \), with a least element, so that \( \mathcal{I}(P) \cong_{\text{homeo}} \mathcal{I}(T) \) where \( \text{Prim}(T) \) is a \( \lor \)-generating set of \( T \).

**Theorem** For any upper semi-lattice \( T \), with a least element, so that \( \text{Prim}(T) \) is \( \lor \)-generating set of \( T \), there is a poset \( P \) so that \( \mathcal{I}(T) \cong_{\text{homeo}} \mathcal{I}(P) \).

**Proof.**

Let \( T \) be an upper semi-lattice, with a least element \( t_0 \), so that \( \text{Prim}(T) \) is \( \lor \)-generating set of \( T \), and set \( P := \{ \downarrow t : t \in \text{Prim}(T) \} \); then \((P, \subseteq)\) is a poset with a least element \( \downarrow \{t_0\} = \{t_0\} \). Define \( \varphi \) from \( \mathcal{I}(T) \) into \( \mathcal{I}(P) \) by \( \varphi(I) = (\downarrow I) \cap P \), where \( \downarrow I := \text{def} \{ J \in \mathcal{I}(T) : J \subseteq I \} \). Again it easy to see that \( \mathcal{I}(T) \cong_{\text{homeo}} \mathcal{I}(P) \).

Let \((T, \leq)\) be a poset with a least element. Previous two theorems give indeed, necessary and sufficient conditions on \( \mathcal{I}(T) \) and \( \mathcal{I}(P) \) to be homeomorphic spaces. We state these conditions in the following corollary.

**Corollary** For any \((\lor\text{-f.g.)-poset } (T, \leq)\), with a least element, the following statements are equivalent.

1. There is a poset \((P, \leq)\), with a least element, so that \( \mathcal{I}(T) \cong_{\text{homeo}} \mathcal{I}(P) \).

2. There is an upper semi-lattice \( T' \), with a least element, so that \( \mathcal{I}(T) \cong_{\text{homeo}} \mathcal{I}(T') \), where \( \text{Prim}(T') \) is a \( \lor \)-generating set of \( T' \).
7 Convex algebras

$2^T$ shall be identified with the power set of $T$ denoted by $\mathcal{P}(T)$. $2^T$ is a Boolean space endowed with Tychonoff’s topology for which \( \langle \mathcal{U}(A, B) : A, B \in [T]^\omega \rangle \), where $\mathcal{U}(A, B) := \{ F \in 2^T : A \subseteq F \text{ and } F \cap B = \emptyset \}$, is a basis for its topology.

Let $(P, <)$ be a poset and set:

$$\text{Convex}(P) := \{ X : \text{is a convex set } (P, <) \}$$

Recall that $X \subseteq P$ is convex whenever $(x, y \in X) \land (x < t < y \rightarrow t \in X)$.

Claim $\text{Convex}(P)$ is a closed subspace of $\{0, 1\}^P$.

Definition

Let $(P, <)$ be a poset. $\text{Clop}(\text{Convex}(P))$ is called the convex algebra over $P$ denoted by $\text{Convalg}(P)$. 
9 Open Problems

For a partially ordered set (poset) \( \langle P, < \rangle \), set \( \dim \langle P, < \rangle = \kappa \) to be the least \( \lambda \) such that \( \langle P, < \rangle \) embeds in \( \lambda \) product of linear orderings \( \langle P, < \rangle \cong \prod_{i<\lambda} L_i \).

**Problem 1.**

Let \( T_n \) be a an upper semi lattice so that \( \dim T_n = n \). Is there an upper semi lattice \( T_{n-1} \) so that \( \dim T_{n-1} = n - 1 \) and \( B(T_n) \cong B(T_{n-1}) \).

**Problem 2.**

How do sub-upper semi lattice algebras of an upper semi lattice algebras look like? E.g., characterize upper semi-lattices \( T \) such that \( B(T) \leq B(\omega_1 \times \omega_1) \).

**Problem 3.**

It is consistent that any Tail algebra \( B(P) \), \( P \) is a poset, embeds in (canonical way) an upper semi lattice algebra?

**Problem 4.**

Is it true that for any uncountable upper semi-lattice \( T \), \( B(T) \) has an uncountable chain or an uncountable anti-chain?

**Problem 5.**

It is consistent that for each \( n \in \mathbb{N} \), there is an upper semi lattice \( T^\lor \) of dimension \( n \) so that: \( inc(B(T^\lor)) = ln(B(T^\lor)) = \aleph_1 \).

**Problem 6.**

Let \( T \) be a poset so that each point, in \( Id(T) \), has a countable local basis and \( Id(T) \) has no bad-points. Is \( B(T) \) isomorphic to an interval algebra?

**Problem 7.**

Let \( B := \text{Int}(C) \) be an interval algebra over the chain \( C \), and \( B_0, B_1 \) subalgebras of \( B := \text{Int}(C) \). Call \( B_0, B_1 \) relatively prime in \( B := \text{Int}(C) \) whenever:
\( B \cong B_0 \times B_1; B_0 \wedge B_1 = \{0, 1\} \).
Is it true that if \( B_0 \) and \( B_1 \) are relatively prime in \( \text{Int}(C) \) they are complements of each other in \( \text{Sub}(\text{Int}(C)) \) and vice versa?
Problem 8.

Characterize $Int(C)$ so that if $D$ is a sub-chain of $Int(C)$ then there is a boolean algebra $E$ such that: $Int(C) \simeq \langle D \rangle \times E$.

Problem 9.

Characterize $P$ so that $B(P) \simeq B(Id(P))$.

Problem 10:

Is it consistent to assume the existence of an interval algebra $|B| = \aleph_1 = inc(B) = d(B)$ so that each of its subalgebra is isomorphic to an interval algebra.

Problem 11:

Is it consistent to assume the existence of an interval $B$, $|B| = \kappa^+$, $inc(B) = \kappa$, $d(B) \neq \aleph_0, \kappa$; so that each of its subalgebra is isomorphic to an interval algebra.

Problem 12:

Is there an uncountable upper semi lattice algebra $B(T)$ which is a hereditarily upper semi-lattice algebra?

Problem 13:

Let $(T, <)$ be a upper semi lattice so that $dim(T) = p < \omega$. Is every subalgebra of $B(T)$ generated by an upper semi lattice $T^*$ so that $dim(T^*) \leq p + 1$?

Problem 14:

Characterize $(T, <)$ so that $B(T)$ is a subalgebra of $B(C_1 \times C_2 \times ... \times C_n)$ where $C_i$s are chains.

Problem 15:

Do semi lattices $(T, <)$ of $dim(T) = 3$ are so that $B(T)$ is a subalgebra of an interval algebra?

Problem 16:

Characterize $T, K$ so that $dim(T) = dim(K) + 1$ and $B(T) \simeq B(K)$.
Problem 17:

Denote by $\text{Convalg}(P)$, $B(T)$, $F(P)$, $\text{Int}(L)$, $B_{pt}(T)$ any convex boolean algebra, algebra generated by upper semi lattice $T$, free algebra over $P$, an interval algebra over $L$, and a pseutree-algebra over $T$ respectively. Characterize:

$\text{Convalg}(P) \cong F(Q)$

$\text{Convalg}(P) \cong B(T)$

$\text{Convalg}(P) \cong B_{pt}(T)$

$\text{Convalg}(P) \cong \text{Int}(L)$

Problem 18:

Let $CPU(B)$ be the class of boolean algebras that are product of three pairwise non isomorphic elements from the classes respectively of convex algebras $C(B)$, pseudotree-algebras $P(B)$, upper semi-lattice algebras $U(B)$, respectively.

Is $CPU(B)$ a new class different from the previously defined ones?

References


