A Game on towers

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June 5, 2010

Definition

A filter $\mathcal{F}$ on ω is *p-filter* if for each $D \in [\mathcal{F}]^\omega$ there exists $p \in \mathcal{F}$ such that $p \subseteq^* d$ for each $d \in D$. The set p is called *pseudointersection* of D. An ideal $\mathcal{I}$, for which the dual filter $\mathcal{I}^*$ is an p-filter, is *p-ideal*.

Lemma (M. Talagrand)

For a filter $\mathcal{F}$ in $P(\omega)$ the following are equivalent:

1. $\mathcal{F}$ *is non-meager subset of* $P(\omega)$.
2. $\mathcal{F}$ *is unbounded (i.e. enumerating functions of sets in* $\mathcal{F}$ *are unbounded subset of* $({}^{\omega}\omega, <^*)$.*)*
3. *For each decomposition of* $\omega = \bigcup I_n$ *into intervals there is a set* $F \in \mathcal{F}$ *such that* $F \cap I_n = \emptyset$ *for infinitely many intervals.*
4. *fin* $\nleq_{RB} \mathcal{F}^*$

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3. *For each decomposition of $\omega = \bigcup I_n$ into intervals there is a set $F \in \mathcal{F}$ such that $F \cap I_n = \emptyset$ for infinitely many intervals.*
4. *$fin \not\leq_{RB} \mathcal{F}^*$*

Definition and Lemma (A. Miller ?)

For a filter $\mathcal{F}$ in $P(\omega)$ the following are equivalent:

1. *$\mathcal{F}$ is rapid.*
2. *Enumerating functions of sets in $\mathcal{F}$ are dominating family.*
3. *For every sequence $\{t_i : t_i \in [\omega]^{<\omega}, i \in \omega\}$ there is $F \in \mathcal{F}$ such that $|F \cap t_i| < i$ for each $i \in \omega$.*

Definition

Forcing P is ${}^{\omega}\omega$ *bounding* if for every P-generic filter G and each $f \in {}^{\omega}\omega \cap V[G]$ there is $g \in {}^{\omega}\omega \cap V$ such that $f < g$.

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Definition

Forcing P has *Sacks property* if for every P-generic filter G and each $f \in {}^{\omega}\omega \cap V[G]$ there is a sequence $\{H_i \in [\omega]^i : i \in \omega\} \in V$ such that $f \in \prod H_i$.

Definition (Grigorieff's forcing)

Let $\mathcal{F}$ be a filter on ω. Put

$$G(\mathcal{F}) = \left(\{p : I \to 2; I \in \mathcal{F}^*\}, \supset\right).$$

The forcing notion $G(\mathcal{F})$ is called *Grigorieff's forcing*.

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Lemma

Let $\mathcal{F}$ be a non-meager p-filter on ω. Then the Grigorieff's forcing $G(\mathcal{F})$ is proper and ${}^{\omega}\omega$ bounding.

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Definition

Let Φ be an automorphism of $P(\omega)/\mathit{fin}$. Φ is *trivial* on $A \subset \omega$ if there is $A' =^* A$ and a 1-to-1 function $f : A' \to \omega$ such that $\Phi([B]) = [f[B]]$ for every $B \subset A$.
$\mathit{triv}(\Phi) = \{A : \Phi \text{ is trivial on } A\}$.

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Theorem

Let Φ be an automorphism of $P(\omega)/fin$. If $\mathcal{F}$ is a non-meager p-filter such that $triv(\Phi) \cap \mathcal{F} = \emptyset$ and if G is $G(\mathcal{F})$-generic, then the family

$$\{\Phi(p^{-1}(1)), \omega \setminus \Phi(p^{-1}(0)) : p \in G\}$$

is an unfilled gap (in $V[G]$).

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Definition (Guided Grigorieff's forcing)

Let $\mathcal{T} = \{T_\alpha : T_\alpha \in [\omega]^\omega, \alpha \in \kappa\}$ be a strictly increasing tower, i.e. $T_\alpha \subset^* T_\beta$ and $|T_\beta \setminus T_\alpha| = \omega$ for $\alpha < \beta < \kappa$.
Denote $\mathcal{A} = \{A_\alpha = T_{\alpha+1} \setminus T_\alpha, \alpha \in \kappa\}$. For $F = \{f_\alpha : A_\alpha \to 2\}$ the forcing notion $P(\mathcal{T}, F)$ consists of pairs (g, β) where $\beta \in \kappa$ and g is a function with domain $D(g) =_* T_\beta$ to 2. Moreover $(g, \beta) \in P(\mathcal{T}, F)$ iff $g \restriction A_\alpha =_* f_\alpha$ for each $\alpha < \beta$.
The ordering is reversed inclusion, $(g, \beta) \leq (h, \gamma)$ iff $h \subseteq g$.

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Lemma

Let $\mathcal{T}$ be a strictly increasing tower which generates a non-meager ideal. Then $P(\mathcal{T}, F)$ is a proper ${}^\omega\omega$ bounding forcing notion for each choice of F.

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This forcing is useful for introducing strong-Q-sequence while keeping $\mathfrak{d}$ small.

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What about Sacks property for Guided Grigorieff's forcing $P(\mathcal{T}, F)$?
What is the condition for $\mathcal{T}$?

Definition (p-filter game $FG(\mathcal{F})$)

Let $\mathcal{F}$ be a filter in $P(\omega)$ containing no finite sets. In n-th move player I plays a filter set $F_n \in \mathcal{F}$ and player II responds with its finite subset $B_n \in [F_n]^{<\omega}$. After ω many moves player II wins if $\bigcup\{B_n : n \in \omega\} \in \mathcal{F}$ and player I wins otherwise.

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Lemma (Laflamme)

$\mathcal{F}$ is non-meager p-filter in $P(\omega)$ if and only if player I has no winning strategy in game $FG(\mathcal{F})$.

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Lemma

Let $\mathcal{F}$ be a rapid p-filter. For each strategy of player I for the p-filter game $FG(\mathcal{F})$ there exists some sequence $\{B_n : |B_n| < n, n \in \omega\}$ of moves of player II which beats this strategy.

Definition (p-filter game)

Let $\mathcal{F}$ be filter in $P(\omega)$ containing no finite sets. The following game is called *p-filter game* $G_{\mathcal{F}}$. In n-th move player I plays a filter set $F_n \in \mathcal{F}$ and player II responds with its finite subset $B_n \in [F_n]^{<\omega}$. After ω many moves player II wins if $\bigcup\{B_n : n \in \omega\} \in \mathcal{F}$ and player I wins otherwise.

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Definition (p-filter game for tower)

In n-th move player I plays an ordinal $\alpha_n < \kappa$ and a finite set $A_n \in [\omega]^{<\omega}$. Player II responds with a finite set B_n, $B_n \cap (T_{\alpha_n} \cup A_n) = \emptyset$.
After ω many moves player II wins if there is some $\alpha < \kappa$ such that $\bigcup\{B_n : n \in \omega\} \cup T_\alpha =^* \omega$ and player I wins otherwise.

Definition (p-filter game for tower)

Let $\mathcal{T} = \{T_\alpha : \alpha \in \kappa\}$ be an increasing tower. In n-th move player I plays an ordinal $\alpha_n < \kappa$ and a finite set $A_n \in [\omega]^{<\omega}$. Player II responds with a finite set B_n, $B_n \cap (T_{\alpha_n} \cup A_n) = \emptyset$. After ω many moves player II wins if there is some $\alpha < \kappa$ such that $\bigcup\{B_n : n \in \omega\} \cup T_\alpha =^* \omega$ and player I wins otherwise.

Definition (Tower game $TG(\mathcal{T})$)

Let $\mathcal{T} = \{T_\alpha : \alpha \in \kappa\}$ be an increasing tower. In n-th move player I plays an ordinal α_n, $\beta_{n-1} < \alpha_n < \kappa$ (where $\beta_{-1} = -1$) and a finite set $A_n \in [\omega]^{<\omega}$. Player II responds with an ordinal $\beta_n < \kappa$ and a finite set B_n, $B_n \cap (T_{\alpha_n} \cup A_n) = \emptyset$. After ω many moves player II wins if $\bigcup\{B_n : n \in \omega\} \cup T_\alpha =^* \omega$ for $\alpha = sup\{\alpha_n : n \in \omega\}$ and player I wins otherwise.

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Tower $\mathcal{T}$ is called *d-tower* if Player I has no winning strategy in the game $TG(\mathcal{T})$.

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Fact

Every d-tower generates non-meager p-ideal.

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If $\mathcal{T}$ is rapid d-tower then filter dual to ideal generated by $\mathcal{T}$ is rapid p-filter.

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$\diamond \Rightarrow$ *there is a rapid d-tower.*