Continuous Cofinal Maps on Ultrafilters

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Def. $\mathcal{U} \subseteq \mathcal{P}(\omega)$ is an *ultrafilter* if

1. $X \in \mathcal{U}$ and $X \subseteq Y$ implies $Y \in \mathcal{U}$;
2. $\forall X, Y \in \mathcal{U}$, $X \cap Y \in \mathcal{U}$;
3. $\forall X \subseteq \omega$, either $X \in \mathcal{U}$ or $\omega \setminus X \in \mathcal{U}$, but not both.

Unless specifically stated otherwise, all ultrafilters in this talk are non-principle.

Def. $\mathcal{V} \leq_T \mathcal{U}$ iff there is a *Tukey* map $g : \mathcal{V} \to \mathcal{U}$ taking unbounded subsets of $\mathcal{V}$ to unbounded subsets of $\mathcal{U}$.

Equivalently, $\mathcal{U} \geq_T \mathcal{V}$ iff there is a *cofinal* map $f : \mathcal{U} \to \mathcal{V}$ taking cofinal subsets of $\mathcal{U}$ to cofinal subsets of $\mathcal{V}$.

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$\mathcal{U} \geq_T \mathcal{V} \Rightarrow$ there is a *monotone* cofinal map witnessing this.

Fact. $\mathcal{U} \geq_T \mathcal{V} \Rightarrow \mathrm{cof}(\mathcal{U}) \geq \mathrm{cof}(\mathcal{V})$ and $\mathrm{add}(\mathcal{U}) \leq \mathrm{add}(\mathcal{V})$.

$\mathcal{U} \equiv_T \mathcal{V}$ iff $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$.

Fact. $\equiv_T$ is an equivalence relation. $\leq_T$ is a partial ordering on the equivalence classes.

Fact. $\mathcal{U} \equiv_T \mathcal{V}$ iff $\mathcal{U}$ and $\mathcal{V}$ are cofinally equivalent.

Motivations

1. A special class of directed systems of size $\mathfrak{c}$.
(In contrast to non-classification theorems of Todorcevic for directed posets of size $\mathfrak{c}$)

2. $\mathcal{U} \geq_{RK} \mathcal{V}$ implies $\mathcal{U} \geq_T \mathcal{V}$.

$\mathcal{U} \geq_{RK} \mathcal{V}$ iff there is a function $h : \omega \to \omega$ such that
$\mathcal{V} = h(\mathcal{U}) := \{X \subseteq \omega : h^{-1}(X) \in \mathcal{U}\}$.

3. Isbell's Problem.

(Much has been done to understand the structure of the Tukey types of ultrafilters by Milovich, Todorcevic, Dobrinen, and Raghavan.)

Thm. [Isbell 65] There is an ultrafilter $\mathcal{U}_{\mathrm{top}} \equiv_T [\mathfrak{c}]^{<\omega}$.

Note: $\mathcal{V} \equiv_T [\mathfrak{c}]^{<\omega}$ iff $\exists S \in [\mathcal{V}]^{\mathfrak{c}}\ \forall T \in [S]^{\omega}\ (\bigcap T \notin \mathcal{V})$.

$$\mathcal{U}_{\mathrm{top}}$$

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Note: There are $2^{\mathfrak{c}}$ different ultrafilters in the Tukey equivalence class of $\mathcal{U}_{\mathrm{top}}$.

Isbell's Problem. [Isbell 65] Is there always (in ZFC) an ultrafilter $\mathcal{U}$ such that $\mathcal{U} <_T \mathcal{U}_{\mathrm{top}}$?

Def. (follows from [Solecki/Todorcevic 04]) An ultrafilter $\mathcal{V}$ is *basic* if each convergent sequence has a bounded subsequence.

Fact. If $\mathcal{U}$ is basic, then $\mathcal{U} <_T [\mathfrak{c}]^{<\omega}$.

Thm. [D/T 10] An ultrafilter is basic iff it is a p-point. Hence, every p-point is not Tukey top.

$\mathcal{U}$ is a *p-point* if for each sequence $X_0 \supseteq X_1 \supseteq \dots$ in $\mathcal{U}$, there is a $Y \in \mathcal{U}$ such that for each $n < \omega$, $Y \subseteq^* X_n$ (i.e. $|Y \setminus X_n| < \omega$).

Note. Fubini products of p-points and more generally, *basically generated* ultrafilters are strictly below the top.

Thm. [D/T 10] Suppose $\mathcal{U}$ is a p-point and $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal map. Then there is an $\tilde{X} \in \mathcal{U}$ such that $f \upharpoonright (\mathcal{U} \upharpoonright \tilde{X})$ is continuous.

Moreover, $f \upharpoonright (\mathcal{U} \upharpoonright \tilde{X})$ can be extended to a continuous monotone map $\tilde{f} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $\tilde{f} : \mathcal{P}(k) \to \mathcal{P}(k)$ for each $k < \omega$.

So the Tukey equivalence class of a p-point has size $\mathfrak{c}$.

Thm. [Raghavan/Todorcevic] If $\mathcal{V}$ is selective and $\mathcal{U} \geq_T \mathcal{V}$ is witnessed by a continuous cofinal map, then $\mathcal{U} \geq_{RK} \mathcal{V}$.

Questions.

1. Are there ultrafilters besides p-points which carry continuous cofinal maps?
2. Does the existence of continuous cofinal maps get inherited downwards in the Tukey ordering?

Certain continuous maps on cofinal subsets can be extended to all of $\mathcal{P}(\omega)$

Thm. [D] Suppose $\mathcal{U}$ is an ultrafilter, $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal map, and there is a cofinal subset $\mathcal{X} \subseteq \mathcal{U}$ such that

1. $f \restriction \mathcal{X}$ is continuous;
2. $f \restriction \mathcal{X}$ is given by a map $\hat{f}$ which is level and initial segment preserving.

Then there is a continuous, monotone $\tilde{f} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $\tilde{f} \restriction \mathcal{X} = f \restriction \mathcal{X}$, $\tilde{f} \restriction \mathcal{U} : \mathcal{U} \to \mathcal{V}$ is a cofinal map, and $\tilde{f}$ is given by a monotone, level and initial segment preserving map.

Rk. Every p-point satisfies these conditions.

A guarantee of T-downward preservation of existence of continuous cofinal maps

Thm. [D] Let $\mathcal{U}$ be an ultrafilter such that whenever $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal function, then there exists an $\mathcal{X} \subseteq \mathcal{U}$ such that

1. $\mathcal{X}$ is cofinal in $\mathcal{U}$;
2. $\mathcal{X}$ is closed under finite intersections, and for each $Z \in \overline{\mathcal{X}}$, there is a decreasing sequence $X_0 \supseteq X_1 \supseteq \ldots \supseteq Z$ such that each $X_n \in \mathcal{X}$ and $\bigcap_{n<\omega} X_n = Z$;
3. $f \restriction \mathcal{X}$ is continuous and given by a level and initial segment preserving monotone map $\hat{f}$.

Then for each $\mathcal{W} \leq_T \mathcal{U}$, if $h : \mathcal{W} \to \mathcal{V}$ is a monotone cofinal map, then there is a cofinal $\mathcal{Y} \subseteq \mathcal{W}$ such that $h \restriction \mathcal{Y}$ is continuous and given by a monotone, level and initial segment preserving map.

Iterations of Fubini products of p-points carry continuous cofinal maps

Thm. [D] Let $\mathcal{U}_*$ be any countable iteration of Fubini products of p-points. If $f : \mathcal{U}_* \to \mathcal{V}$ is a monotone cofinal function, then there is a $\mathcal{U}_*$ tree $\tilde{T}$ such that f is continuous on the collection of $\mathcal{U}_*$ trees $T \subseteq \tilde{T}$ and is given by an initial segment and level preserving function.

Cor. [D] If $\mathcal{U}_*$ is a countable iteration of Fubini products of p-points and $\mathcal{W} \leq_T \mathcal{U}_*$, then for every ultrafilter $\mathcal{V} \leq_T \mathcal{W}$, there is a continuous monotone cofinal map $f : \mathcal{W} \to \mathcal{V}$.

Ultrafilters on FIN

$\text{FIN} = [\omega]^{<\omega} \setminus \{\emptyset\}$.

Def. An (idempotent) ultrafilter $\mathcal{U}$ on FIN is *ordered union* if it is generated by sets of the form $[X]$ where X is an infinite block-sequence. ($[X]$ is the collection of all finite unions of members of X.) (In [D/T 10] we called these *block-generated* ultrafilters.)

Thm. [Blass 87] If $\mathcal{U}$ is an ordered union ultrafilter on FIN, then both $\mathcal{U}_{\text{min}}$ and $\mathcal{U}_{\text{max}}$ are selective.

Fact. If $\mathcal{U}$ is a ordered union ultrafilter on FIN, then $\mathcal{U}_{\text{min,max}}$ is rapid but neither a p-point nor a q-point.

Def. An ordered union ultrafilter $\mathcal{U}$ is *block-basic* if whenever we are given a sequence (X_n) of infinite block sequences in FIN such that each $[X_n] \in \mathcal{U}$ and (X_n) converges to some infinite block sequence X such that $[X] \in \mathcal{U}$, then there is an infinite subsequence (X_{n_k}) such that $\bigcap_{k<\omega}[X_{n_k}] \in \mathcal{U}$.

Thm. [Blass 87 and D/T 10] The following are equivalent for an ordered union ultrafilter $\mathcal{U}$ on FIN.

1. $\mathcal{U}$ is block-basic.

2. $\mathcal{U}$ is stable ordered-union. (For every sequence (X_n) of infinite block sequences of FIN such that $[X_n] \in \mathcal{U}$ and $X_{n+1} \leq^* X_n$ for each n, there is an infinite block sequence X such that $[X] \in \mathcal{U}$ and $X \leq^* X_n$ for each n.)

3. $\mathcal{U}$ has the 2-dimensional Ramsey property.

4. $\mathcal{U}$ has the Ramsey property.

Continuous cofinal maps

Thm. [D/T 10] Suppose $\mathcal{U}$ is a block-basic ultrafilter on FIN and that $\mathcal{U} \geq_T \mathcal{V}$ for some ultrafilter $\mathcal{V}$ on any countable index set I. Then there is a monotone continuous map $f : \mathcal{P}(\text{FIN}) \to \mathcal{P}(I)$ such that $f''\mathcal{U}$ is a cofinal subset of $\mathcal{V}$.

Thm. [D/T 10] Suppose $\mathcal{U}$ is a block-basic ultrafilter on FIN and $\mathcal{V}$ is any ultrafilter on a countable index set I. If $\mathcal{U}_{\text{min,max}} \geq_T \mathcal{V}$, then there are an infinite block sequence $\tilde{X}$ such that $[\tilde{X}] \in \mathcal{U}$ and a monotone continuous function f from $\{[X]_{\text{min,max}} : X \leq \tilde{X}\}$ into $\mathcal{P}(I)$ whose restriction to $\{[X]_{\text{min,max}} : X \leq \tilde{X},\ [X] \in \mathcal{U}\}$ has cofinal range in $\mathcal{V}$.

Using the second theorem from the previous slide, one obtains the following.

Thm. [D/T 10] Assuming CH, there is a block-basic ultrafilter $\mathcal{U}$ on FIN such that $\mathcal{U}_{\text{min,max}} <_T \mathcal{U}$ and $\mathcal{U}_{\text{min}}$ and $\mathcal{U}_{\text{max}}$ are Tukey incomparable.

$$\begin{tikzcd}
& \mathcal{U} \arrow[d] & \\
& \mathcal{U}_{\text{min,max}} \arrow[dl] \arrow[dr] & \\
\mathcal{U}_{\text{min}} \arrow[dr] & & \mathcal{U}_{\text{max}} \arrow[dl] \\
& \mathbf{0} &
\end{tikzcd}$$

A characterization of ultrafilters $\mathcal{U} <_T \mathcal{U}_{\text{top}}$.

Prop. [Milovich 09] There is an ultrafilter $\mathcal{U}$ such that $(\mathcal{U}, \supseteq) <_T \mathcal{U}_{\text{top}}$ iff there is an ultrafilter $\mathcal{V}$ such that $(\mathcal{V}, \supseteq^*) <_T \mathcal{U}_{\text{top}}$.

Assume $\neg$CH. Then $\diamondsuit^-_{[[\omega]^\omega]^\omega}$ holds.

Def. [Todorcevic] $\diamondsuit^-_{[[\omega]^\omega]^\omega}$: There exist ordered pairs $(\mathcal{U}_A, \mathcal{X}_A)$, where $A \in [[\omega]^\omega]^\omega$ and $\mathcal{X}_A \subseteq \mathcal{U}_A \subseteq A$, such that for each pair $(\mathcal{U}, \mathcal{X})$ with $\mathcal{X} \subseteq \mathcal{U}$ and $\mathcal{X}, \mathcal{U} \in [[\omega]^\omega]^{\mathfrak{c}}$, $\{A \in [[\omega]^\omega]^\omega : \mathcal{U}_A = \mathcal{U} \cap A, \mathcal{X}_A = \mathcal{X} \cap A\}$ is stationary in $[[\omega]^\omega]^\omega$.

Let $P_A = \{W \in [\omega]^\omega : \exists X \in \mathcal{U}_A (W \cap X = \emptyset)\}$,
$Q_A = \{W \in [\omega]^\omega : \exists (B_n)_{n<\omega} \subseteq \mathcal{X}_A (\forall n < \omega,\ W \subseteq^* B_n)\}$, and
$D_A = P_A \cup Q_A$.
Then for each $A \in [\omega]^\omega$, D_A is dense open in $[\omega]^\omega$.

Fact. [D/T 10] For any ultrafilter $\mathcal{U}$, $\{A \in [[\omega]^\omega]^\omega : \mathcal{U} \cap D_A \neq \emptyset\}$ is stationary.

Thm. [D/T 10] If $\mathcal{U} \cap D_A \neq \emptyset$ for club many $A \in [\omega]^\omega$, then $\mathcal{U} <_T \mathcal{U}_{\text{top}}$.

Thm. [D/T 10] If $\mathcal{U}$ is a p-point, then $\mathcal{U} \cap D_A \neq \emptyset$ for all $A \in [[\omega]^\omega]^\omega$.

Let $P'_A = \{W \in [\omega]^\omega : \forall X \in \mathcal{X}_A (W \subseteq^* X)\}$. Let $D'_A = P'_A \cup Q_A$. By the same proof as for D_A, we see that D'_A is dense open in $[\omega]^\omega$.

Fact. [D/T 10] If $\mathcal{U} \cap D'_A \neq \emptyset$ for club many A, then $\mathcal{U}$ is a p-point.

Some Conjectures

Conjecture. If $\mathcal{U} <_T \mathcal{U}_{\text{top}}$, then $\mathcal{U}$ supports continuous cofinal maps.

Conjecture. Suppose there is a supercompact cardinal. If $\mathcal{U}$ is selective, then there are exactly 2 Tukey types of ultrafilters in $L(\mathbb{R})[\mathcal{U}]$.

Conjecture. Suppose there is a supercompact cardinal. If $\mathcal{U}$ is block-basic, then there are exactly 5 Tukey types of ultrafilters in $L(\mathbb{R})[\mathcal{U}]$.