The Logic of Stone Spaces

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**Basics**

\( \text{CL} = \text{the variety of all closure algebras} \ (B, \ C) \)

\( X^* = (\mathcal{P}X, C) \) where \( X \) is a topological space

View subvarieties of \( \text{CL} \) as extensions of Lewis’ \( \text{S4} \)

- \( \text{S4} \leftrightarrow \text{CL} \)
- \( \text{S4.1} \leftrightarrow \text{CL} + \text{IC}_x \leq \text{Cl}_x \)
- \( \text{S4.2} \leftrightarrow \text{CL} + \text{Cl}_x \leq \text{IC}_x \)
- etc.

**Theorem (McKinsey-Tarski)** If \( X \) is metrizable and has no isolated points, then \( X^* \) generates \( \text{CL} \).
Aim

For a Boolean algebra $B$ with Stone space $X$, to determine the subvariety of $\text{CL}$ generated by $X^*$, i.e. the modal logic of $X$. We can do this if $B$ is complete or if $B$ is countable.

Note For $B$ countable and free, $X$ is the Cantor space, so by the McKinsey-Tarski theorem its logic is $\text{S4}$. 
Tools

Each quasiorder $Q$ is a topological space where opens $\equiv$ upsets.

Many subvarieties of $\textbf{CL}$ are generated by classes of quasiorders.

- **S4** by finite quasitrees.
- **S4.1** by finite quasitrees with top level simple nodes.
- **S4.2** by the $Q \oplus C$ with $Q$ finite quasitree and $C$ cluster.

\[ X \xrightarrow{f} Y \quad \text{cont + open + onto} \quad \Rightarrow \quad Y^* \xleftarrow{f^{-1}} X^* \quad \text{CL-embedding.} \]
Tools

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- $\text{S4}$ by finite quasitrees.
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$$X \xrightarrow{f} Y \text{ cont + open + onto } \implies Y^* \xleftarrow{f^{-1}} X^* \text{ CL-embedding.}$$

Example  To show the logic of $X$ is $\text{S4}$

Enough to find an onto interior $X \xrightarrow{f} Q$ for each finite quasitree $Q$ as $X^*$ will contain a generating set for $\text{S4}$. 
Our job amounts to finding interior onto maps $X \xrightarrow{f} Q$.

Let's look at some easy examples ...
Easiest example

For $X$ the Stone space of $B$, when is there an interior onto map

When $X$ has a proper dense open set $U = f^{-1}[\text{top}]$.
When $B$ has a proper ideal whose join is 1.
When $B$ is infinite.
Easiest example

For $X$ the Stone space of $B$, when is there an interior onto map $f$?

When $X$ has a proper dense open set $U$ ($= f^{-1}[\text{top}]$).
When $B$ has a proper ideal whose join is 1.
When $B$ is infinite.
Next easiest example

For $X$ the Stone space of $B$, when is there an interior onto map
Next easiest example

For $X$ the Stone space of $B$, when is there an interior onto map $f$?

When $X$ has disjoint regular open $U, V$ with $U \cup V$ proper dense.
When $B$ has a non-principal normal ideal.
When $B$ is incomplete.
The logic of $\omega^*$

$\beta\omega = \text{the Stone Cech compactification of } \omega$

$\omega^* = \text{the remainder } \beta\omega - \omega$

$\omega^* = \text{the Stone space of } \mathcal{P}\omega / \text{Fin}$.

Theorem  The logic of $\omega^*$ is $\textbf{S4}$.

Proof.  We need an interior onto map $\omega^* \xrightarrow{f} Q$ for each finite quasitree $Q$. For this we need a technical result to recursively build a tree of ideals in our Boolean algebra.
Lemma \((a = 2^\omega)\). For \(P\) a partition of \(b \in \mathcal{P}_\omega/\text{Fin}\) and \(m \geq 1\), there are sets \(P_1, \ldots, P_m\) and maps \(f_1, \ldots, f_m\) with

1. \(P_1 \cup \cdots \cup P_m = P\) and \(P_i \cap P_j = \emptyset\) for each \(i \neq j\).
2. \(f_i : \text{Infinite}(P) \to P_i\) is 1-1 for each \(i \leq m\).
3. \(f_i(c) \in \text{Support}_P(c)\) for each \(c \in \text{Infinite}(P)\) and each \(i \leq m\).

Note \((a = 2^\omega)\) is an additional assumption of set theory.

Note We use this to recursively build a tree of ideals.
Corollaries

Theorem  The logic of $\beta\omega$ is $\textbf{S4.1.2}$.

Proof. Any interior $\omega^* \to Q$ lifts to an interior $\beta\omega \to Q \oplus 1$ and this is exactly what we need.

Theorem  For $B$ a complete Boolean algebra with Stone space $X$.

1. If $B$ is finite, the logic of $X$ is classical.
2. If $B$ is infinite and atomic, the logic of $X$ is $\textbf{S4.1.2}$.
3. Otherwise the logic of $X$ is $\textbf{S4.2}$.

Proof. Such $X$ has a closed subspace homeomorphic to $\beta\omega$. We use this to build our map $X \to Q \oplus C$ for the difficult case 3.
Countable Boolean algebras

For $B$ Boolean with Stone space $X$ the following are equivalent

- $B$ is countable
- $B$ is generated by a countable chain $C$
- $X$ is metrizable

The atomless case gives $S4$ by McKinsey-Tarski.

The scattered case gives $\text{Grz}_n$ for some $n \leq \omega$ by old results.

So we may assume $B$ is generated by a chain $C$ where each interval contains a cover, and the condensation $D$ of $C$ is $\mathbb{Q}$. We will show $S4.1$ is the logic in this case.
Our setup ...

\[ D = \text{condensation of } C \]
\[ Y = \text{Stone space of free Boolean ext of } D \text{ (so } Y \cong \text{Cantor)} \]
\[ Y \leq X \]

Let's sketch the idea ...

We get this as \( Y \cong \text{Cantor} \)
The hard part is to use the way \( Y \) sits in \( X \) to extend to ...

\[
\begin{array}{ccc}
X & \xrightarrow{g} & \text{Diagram 1} \\
\end{array}
\]

As squishing the top parts is interior we get

\[
\begin{array}{ccc}
X & \xrightarrow{h} & \text{Diagram 2} \\
\end{array}
\]

The \( Q \) we can get on the right are the ones we need to show S4.1.
Questions

Is the assumption \((a = 2^{\omega})\) necessary for the \(\omega^*\) result?

Extend countable results to any \(B\) generated by a chain, or tree.

**Conjecture**

The varieties generated by \(X^*\) for a Stone space \(X\) are exactly the finite joins of the ones above.

**Little question**

Does every atomless \(B\) have a dense ideal \(I\) with \(B/I\) atomless?