Relation algebras as expanded FL-algebras

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Outline

- Relation algebras as residuated Boolean monoids
- Expansions of FL-algebras
- Quasi relation algebras
- Decidability
- Conclusion and open problems

Relation algebras

Definition (Tarski 1941)

Relation algebras are algebras $(A, \wedge, \vee, ', \bot, \top, \cdot, ^\smile, 1)$ such that

- $(A, \wedge, \vee, ', \bot, \top)$ is a Boolean algebra
- $(A, \cdot, 1)$ is a monoid and
- for all $x, y, z \in A$, $\quad (x \vee y)z = xz \vee yz \qquad (x + y)^\smile = x^\smile + y^\smile$

  $x^{\smile\smile} = x \qquad (xy)^\smile = y^\smile x^\smile \qquad x^\smile (xy)' \leq y'$

The five identities are equivalent to

$$xy \leq z' \iff x^\smile z \leq y' \iff zy^\smile \leq x'$$

so defining *conjugates* $x \triangleright z = x^\smile z$ and $z \triangleleft y = zy^\smile$ we have

$$xy \leq z' \iff x \triangleright z \leq y' \iff z \triangleleft y \leq x'$$

Residuated Boolean monoids

Definition (Birkhoff 1948, Jónsson 1991)

Residuated Boolean monoids are algebras $(A, \wedge, \vee, ', \bot, \top, \cdot, \triangleright, \triangleleft, 1)$ s. t.

- $(A, \wedge, \vee, ', \bot, \top)$ is a Boolean algebra
- $(A, \cdot, 1)$ is a monoid and
- for all $x, y, z \in A$, $\quad xy \le z' \iff x \triangleright z \le y' \iff z \triangleleft y \le x'$

Examples: For any monoid $\mathbf{M} = (M, *, e)$ the powerset monoid $\mathcal{P}(\mathbf{M}) = (\mathcal{P}(M), \cap, \cup, ', \emptyset, M, \cdot, \triangleright, \triangleleft, \{e\})$ is a residuated Boolean monoid

where $XY = \{x * y : x \in X, y \in Y\}$,
$X \triangleright Y = \{z : x * z = y, x \in X, y \in Y\}$,
$X \triangleleft Y = \{z : z * y = x, x \in X, y \in Y\}$

If $\mathbf{G} = (G, *, ^{-1})$ is a group, $\mathcal{P}(\mathbf{G})$ is a relation algebra, $X^{\smile} = \{x^{-1}{:}x{\in}X\}$

RM = the variety of residuated Boolean monoids

RA = the variety of relation algebras

Theorem (Jónsson and Tsinakis 1993)

RA *is termequivalent to the subvariety of* **RM** *defined by* $(x \triangleright y)z = x \triangleright (yz)$

The termequivalence is given by $x \triangleright y = x^{\smile} y$, $x \triangleleft y = xy^{\smile}$ *and* $x^{\smile} = x \triangleright e$

Aim to lift this result to residuated lattices and FL-algebras

RA and **RM** have undecidable equational theories

Want to find a larger variety "close to" RA that has a decidable equational theory, but ...

Kurucz, Nemeti, Sain and Simon [1993] proved that the variety of all Boolean algebras with an associative operator, as well as a "large number" of expanded subvarieties have undecidable equational theories

Residuals

The conjugation condition

$$xy \leq z' \iff x \triangleright z \leq y' \iff z \triangleleft y \leq x'$$

can be rewritten (replacing z by z') as

$$xy \leq z \iff y \leq (x \triangleright z')' \iff x \leq (z' \triangleleft y)'$$

so defining *residuals* $x \backslash z = (x \triangleright z')'$ and $z/y = (z' \triangleleft y)'$ get the equivalent *residuation property*

$$xy \leq z \iff y \leq x \backslash z \iff x \leq z/y$$

(this justifies the name *residuated* Boolean monoids)

FL-algebras

Definition (Ono 1990)

A *Full Lambek* (or *FL-*)*algebra* is of the form $(A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ where

- $(A, \wedge, \vee)$ is a lattice
- $(A, \cdot, 1)$ is a monoid
- 0 is a constant (with no properties assumed about it) and
- the *residuation property* holds, i. e., for all $x, y, z \in A$

$$x \cdot y \leq z \quad \iff \quad x \leq z/y \quad \iff \quad y \leq x\backslash z$$

Examples: Complementation free reducts of residuated Boolean monoids

Symmetric $(x^{\smile} = x)$ *relation algebras* with $0 = 1'$, $x\backslash y = (xy')'$ and $x/y = (x'y)'$

In this case $x' = x\backslash 0 = 0/x$, but for RA in general $x\backslash 0 = (x^{\smile}1'')' = x^{\smile\prime}$ so complementation is not recovered by this term

In an FL-algebra there are two *linear negations*

$$-x = 0/x \qquad\qquad \sim x = x\backslash 0$$

but they need not coincide or be involutive

To interpret relation algebras into FL-algebras we expand FL-algebras with a unary operation:

Definition

An *FL$'$-algebra* is an expansion of an FL-algebra with a unary operation $'$ that satisfies $x'' = x$. Also define the following terms:

- *converses* $x^{\cup} = (\sim x)'$ and $x^{\sqcup} = (-x)'$,
- *conjugates* $x \rhd y = (x\backslash y')'$ and $y \lhd x = (y'/x)'$

and consider the identities

(In) $\sim -x = x = -\sim x$ (involutive law)

(Cy) $\sim x = -x$ (cyclic law)

(Dm) $(x \wedge y)' = x' \vee y'$ (De Morgan, equivalent to $(x \vee y)' = x' \wedge y'$)

Properties of FL′-algebras

Proposition

In an FL′-algebra the following properties hold:

1. $(xy) \triangleright z = y \triangleright (x \triangleright z)$ *and* $z \triangleleft (yx) = (z \triangleleft x) \triangleleft y$
2. $(xy)^{\cup} = y \triangleright x^{\cup}$ *and* $(xy)^{\sqcup} = y^{\sqcup} \triangleleft x$
3. $1 \triangleright x = x$ *and* $x \triangleleft 1 = x$
4. ${\sim}x = -x$ *iff* $x^{\cup} = x^{\sqcup}$ *(cyclic/balanced)*

If (Dm) $(x \wedge y)' = x' \vee y'$ *is assumed then we also have*

- $xy \leq z' \Leftrightarrow x \triangleright z \leq y' \Leftrightarrow z \triangleleft y \leq x'$ *(conjugation)*
- $(x \vee y)^{\cup} = x^{\cup} \vee y^{\cup}$ *and* $(x \vee y)^{\sqcup} = x^{\sqcup} \vee y^{\sqcup}$
- $(x \vee y) \triangleright z = (x \triangleright z) \vee (y \triangleright z)$ *and* $z \triangleleft (x \vee y) = (z \triangleleft x) \vee (z \triangleleft y)$
- $(x \vee y) \triangleleft z = (x \triangleleft z) \vee (y \triangleleft z)$ *and* $z \triangleright (x \vee y) = (z \triangleright x) \vee (z \triangleright y)$

RL′-algebras

FL-algebras are a subvariety of FL′-algebras if we define $x' = x$

Residuated lattices (**RL**) are a subvariety of **FL** if we define $0 = 1$

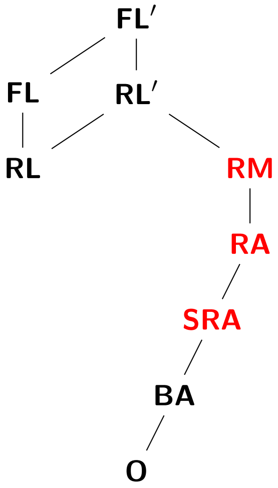
RL′ is the subvariety of **FL′** defined by $1' = 0$

Lemma

In an RL′-algebra the following properties hold:

- $x \triangleright 1 = x^{\cup}$ *and* $1 \triangleleft x = x^{\sqcup}$
- $1^{\cup} = 1^{\sqcup} = 1$

Some subvarieties of $\mathbf{FL}'$



When negation commutes with $'$

Proposition

In an FL'-algebra the following are equivalent:

(Ci) $\sim(x') = (\sim x)'$ *and* $-(x') = (-x)'$ *(commuting involution)*

(ii) $x^{\cup\prime} = x^{\prime\cup}$ *and* $x^{\sqcup\prime} = x^{\prime\sqcup}$ *(commuting converses involution)*

(iii) $x^{\cup\sqcup} = x = x^{\sqcup\cup}$ *(converse involutive)*

(iv) $-x^{\cup} = x' = \sim x^{\sqcup}$

Moreover, each of these properties implies the following identity:

(In) $\sim -x = x = -\sim x$

Quasi relation algebras

Define the term $x + y = {\sim}(-y \cdot -x)$ $\quad (= -({\sim}y \cdot {\sim}x)$ if (In) is assumed)

Proposition

In every InFL$'$-algebra the following are equivalent and they imply $0 = 1'$

1. $(xy)^{\cup} = y^{\cup}x^{\cup}$
2. $(xy)^{\sqcup} = y^{\sqcup}x^{\sqcup}$
3. $x \triangleright y = x^{\cup}y$
4. $y \triangleleft x = yx^{\sqcup}$
5. $(xy)' = x' + y'$

A *quasi relation algebra* (*qRA*) is a CiDmFL$'$-algebra that satisfies $(xy)' = x' + y'$

Lemma

Every qRA is cyclic, i.e., satisfies ${\sim}x = -x$

Examples of quasi relation algebras

Let $G = \text{Aut}(C)$ be the ℓ-group of all order-automorphisms of a chain C, and assume that C has a dual automorphism $\partial : C \to C$

G is a cyclic involutive FL-algebra with $\sim x = -x = x^{-1}$, $x + y = xy$, and $0 = 1$

For $g \in G$, define $g'(x) = g(x^\partial)^\partial$. Then $g'' = g$, 1'=1

$$y = g^{-1\prime}(x) \quad \Leftrightarrow \quad y = g^{-1}(x^\partial)^\partial \quad \Leftrightarrow \quad y^\partial = g^{-1}(x^\partial)$$
$$g(y^\partial)^\partial = x \quad \Leftrightarrow \quad g'(y) = x \quad \Leftrightarrow \quad y = g'^{-1}(x)$$

$(g \vee h)'(x) = (g(x^\partial) \vee h(x^\partial))^\partial = g(x^\partial)^\partial \wedge h(x^\partial)^\partial = (g' \wedge h')(x)$ and

$(gh)'(x) = (g(h(x^\partial)))^\partial = g(h(x^\partial)^{\partial\partial})^\partial = (g'h')(x) = (g' + h')(x)$.

Hence G expanded with $'$ is a quasi relation algebra.

For InFL-algebra $(A, \wedge, \vee, \cdot, \sim, -, 1, 0)$ define $\mathbf{A}^\partial = (A, \vee, \wedge, +, -, \sim, 0, 1)$

$\mathbf{A}^\partial$ is also an InFL-algebra called the *dual* of $\mathbf{A}$

Define $F : \mathbf{InFL} \to \mathbf{InFL}'$ by $F(\mathbf{A}) = \mathbf{A} \times \mathbf{A}^\partial$ expanded with $(a, b)' = (b, a)$

For a homomorphism $h : \mathbf{A} \to \mathbf{B}$ define $F(h) : F(\mathbf{A}) \to F(\mathbf{B})$ by $F(h)(a, b) = (h(a), h(b))$.

Theorem (generalization of Brzozowski 2001)

F is a functor from $\mathbf{InFL}$ *to* $\mathbf{InRL}'$, *and the restriction to cyclic InFL-algebras maps into* $\mathbf{qRA}$.
If G is the reduct functor from $\mathbf{InRL}'$ *to* $\mathbf{InFL}$ *then for any qRA* $\mathbf{C}$, *the map* $\sigma_{\mathbf{C}} : \mathbf{C} \to FG(\mathbf{C})$ *given by* $\sigma_{\mathbf{C}}(a) = (a, a')$ *is an embedding.*

Corollary

The equational theory of $\mathbf{qRA}$ *is a conservative extension of that of* $\mathbf{CyInFL}$; *i.e., every equation over the language of* $\mathbf{CyInFL}$ *that holds in* $\mathbf{qRA}$, *already holds in* $\mathbf{CyInFL}$.

Lifting the Jónsson-Tsinakis result to qRAs

Theorem

qRA *is termequivalent to the subvariety of* $\mathbf{CiDmRL}'$ *defined by* $(x \triangleright y)z = x \triangleright (yz)$

The termequivalence is given by $x \triangleright y = x^{\cup} y$, $x \triangleleft y = xy^{\cup}$ *and* $x^{\cup} = x \triangleright 1$

We also note that to get from **qRA** to **RA** it suffices to add

distributivity: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and

complementation: $x \wedge x' = \bot \quad (= 1 \wedge 1')$ and $x \vee x' = \top \quad (= 1 \vee 1')$

qRAs have a decidable equational theory

We make use of the following result:

Theorem (Yetter 1990, Wille 2005)

The variety **CyInFL** *has a decidable equational theory*

For an **InFL**-term t, we define the *dual* term t^∂ inductively by

$$\begin{array}{rcl} x^\partial = x & & (s \wedge t)^\partial = s^\partial \vee t^\partial \\ 0^\partial = 1 & & (s \vee t)^\partial = s^\partial \wedge t^\partial \\ 1^\partial = 0 & & (s \cdot t)^\partial = s^\partial + t^\partial \\ ({\sim}s)^\partial = -s^\partial & & (s + t)^\partial = s^\partial \cdot t^\partial \\ (-s)^\partial = {\sim}s^\partial & & \end{array}$$

We also define $(s = t)^\partial$ to be $s^\partial = t^\partial$.

Lemma

An equation ε is valid in **InFL** *iff ε^∂ is also valid in* **InFL**.

We fix a partition of the denumerable set of variables into two denuberable sets X and $X^\bullet$, and fix a bijection $x \mapsto x^\bullet$ from the first set to the second (hence $x^{\bullet\bullet}$ denotes x).

For a **qRA**-term t, we define the term t° inductively by

$$\begin{array}{cc}
x^\circ = x & (s'')^\circ = s \\
0^\circ = 0, \quad 1^\circ = 1, & ((s \wedge t)')^\circ = s'^\circ \vee t'^\circ, \\
(0')^\circ = 1, \quad (1')^\circ = 0, & ((s \vee t)')^\circ = s'^\circ \wedge t'^\circ, \\
(s \diamond t)^\circ = s^\circ \diamond t^\circ, \text{ for all } \diamond \in \{\wedge, \vee, \cdot, +\}, & ((s \cdot t)')^\circ = s'^\circ + t'^\circ, \\
(\sim s)^\circ = \sim s^\circ, \quad (-s)^\circ = -s^\circ, & ((s + t)')^\circ = s'^\circ \cdot t'^\circ, \\
((\sim s)')^\circ = -(s'^\circ), \quad ((-s)')^\circ = -(s'^\circ), & (x')^\circ = x^\bullet
\end{array}$$

Lemma

For every **qRA**-*term* t, $t^{\circ\partial}(x_1, \ldots, x_n) = t'^{\circ}(x_1^\bullet, \ldots, x_n^\bullet)$.

For a substitution σ, we define a substitution σ° by $\sigma^\circ(x) = (\sigma(x))^\circ$, if $x \in X$, and $\sigma^\circ(x) = (\sigma(x)')^\circ$, if $x \in X^\bullet$.

Lemma

For every **qRA**-*term* t *and* **qRA**-*substitution* σ, $(\sigma(t))^\circ = \sigma^\circ(t^\circ)$.

Theorem

An equation ε *over* X *holds in* **qRA** *iff the equation* ε° *holds in* **CyInFL**.

Corollary

The equational theory of **qRA** *is decidable.*

Conclusion

By expanding FL-algebras with a unary De Morgan operation one can interpret relation algebras with FL'-algebras

This leads to the variety of quasi relation algebras that has many properties in common with RA

In addition **qRA** has a decidable equational theory

Problem: Is **qRA** generated by its finite members?

Problem: Does the subvariety of distributive qRAs have a decidable equational theory?

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