INTERPOLATION OF κ-COMPACTNESS AND PCF

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Outline

- Basic definitions and results on κ-compactness
- Interpolation results for κ-compactness, using scales from PCF-theory
- Applications to uncountable compactness

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κ-COMPACTNESS

x is a complete accumulation point (CAP) of $A \subset X$ iff for every neighbourhood U of x we have $|U \cap A| = |A|$.
We denote the set of all CAP's of A by A°.

Alexandrov-Urysohn (1920's) : A space is compact iff every infinite subset has a CAP.

DEFINITION. A space κ-compact if every subset of cardinality κ has a CAP.

EXTRAPOLATION : Assume κ is singular and $\kappa_\alpha \nearrow \kappa$ for $\alpha < \mathrm{cf}(\kappa)$. If X is both κ_α-compact for all $\alpha < \mathrm{cf}(\kappa)$ and $\mathrm{cf}(\kappa)$-compact then X is κ-compact.

COROLLARY. A space is compact iff every infinite subset of regular cardinality has a CAP.

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Non-attributed results below are joint with Z. Szentmiklóssy.

INTERPOLATION : $\mu < \kappa < \lambda$ and we deduce κ-compactness of a space X from its μ- and λ-compactness.

DEFINITION. $\Phi(\mu, \kappa, \lambda)$ is the statement: $\mu < \kappa < \lambda = \mathrm{cf}(\lambda)$ and there is $\{S_\xi : \xi < \lambda\} \subset [\kappa]^\mu$ s.t. $A \in [\kappa]^{<\kappa}$ implies $\big|\{\xi : |S_\xi \cap A| = \mu\}\big| < \lambda$.

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If $\Phi(\mu, \kappa, \lambda)$ holds and X is both μ-compact and λ-compact then X is κ-compact.

Proof. Let $Y \in [X]^\kappa$ and $\{S_\xi : \xi < \lambda\} \subset [Y]^\mu$ witness $\Phi(\mu, \kappa, \lambda)$. Pick $p_\xi \in {S_\xi}^\circ$ for all $\xi < \lambda$. There is $p \in X$ s.t. for every nbhd U of p, $\big|\{\xi : |S_\xi \cap U| = \mu\}\big| = \lambda$. By $\Phi(\mu, \kappa, \lambda)$, then $|Y \cap U| = \kappa$, hence $p \in Y^\circ$.

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(i) If $\Phi(\mu, \kappa, \lambda)$ holds then $\mathrm{cf}(\mu) = \mathrm{cf}(\kappa)$, hence κ is singular.
(ii) $\Phi(\mathrm{cf}(\kappa), \kappa, \lambda)$ implies $\Phi(\mu, \kappa, \lambda)$ whenever $\mu < \kappa$ with $\mathrm{cf}(\mu) = \mathrm{cf}(\kappa)$.

DEFINITION. If κ is singular and $\kappa_\alpha \nearrow \kappa$ for $\alpha < \mathrm{cf}(\kappa)$,

$$\{f_\xi : \xi < \lambda\} \subset \prod\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$$

is a scale if it is increasing and cofinal w.r.t. eventual dominance $<^*$.

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If there is a scale of length $\lambda = \mathrm{cf}(\lambda)$ in $\prod\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ then $\Phi(\mathrm{cf}(\kappa), \kappa, \lambda)$ holds.

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Scales play a central role in Shelah's PCF theory $\equiv$ study of products of "progressive" sets of regular cardinals. $A \subset \mathrm{REG}$ is *progressive* iff $|A| < \min A$.

THEOREM (Shelah)

For every singular cardinal κ there are regular cardinals $\kappa_\alpha \nearrow \kappa$ for $\alpha < \mathrm{cf}(\kappa)$ s.t. $\prod\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ has a scale of length κ^+.

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If κ is singular and $\mu < \kappa$ with $\mathrm{cf}(\mu) = \mathrm{cf}(\kappa)$ then $\Phi(\mu, \kappa, \kappa^+)$ holds. So, if X is μ-compact and κ^+-compact then it is κ-compact.

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Arhangel'skii (2008) :
A space is uncountably compact (UCC) iff it is κ-compact for every uncountable κ. Every UCC space is Lindelöf.
Example: one-point "Lindelöfication" of any (uncountable) discrete space.

NOTE. X is linearly Lindelöf (LL) iff it is κ-compact for every uncountable regular κ. By extrapolation, then X is κ-compact whenever $\mathrm{cf}(\kappa) > \omega$. The question when LL implies Lindelöf is an interesting and important question.

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Arhangel'skii (2008) :
A space is uncountably compact (UCC) iff it is κ-compact for every uncountable κ. Every UCC space is Lindelöf.
Example: one-point "Lindelöfication" of any (uncountable) discrete space.

NOTE. X is linearly Lindelöf (LL) iff it is κ-compact for every uncountable regular κ. By extrapolation, then X is κ-compact whenever $\mathrm{cf}(\kappa) > \omega$. The question when LL implies Lindelöf is an interesting and important question.

THEOREM

Any LL and $\aleph_\omega$-compact space is UCC, hence Lindelöf.

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DEFINITION. X is κ-concentrated on $Y \subset X$ iff for every open $U \supset Y$ we have $|X \setminus U| < \kappa$.

THEOREM

If X is κ-concentrated on a compact subset then X is λ-compact for all $\lambda \geq \kappa$.

THEOREM (Arhangel'skii)

Every UCC T_3 space is $\aleph_\omega$-concentrated on a compact subset.

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Every UCC T_1 space X with the *wD* property is $\aleph_\omega$-concentrated on a compact subset.

NOTE. Lindelöf T_3 spaces are normal, while *wD* is a very weak normality property.

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Proof

Set

$$C = X \setminus \cup\{U : U \text{ open }, |U| < \aleph_\omega\}.$$

C is LL, hence compact if countably (i.e. ω-)compact. Otherwise, by wD, there is a discrete collection $\{U_n : n \in \omega\}$ of open sets s.t. $C \cap U_n \neq \emptyset$, hence $|U_n| \geq \aleph_\omega$ for each $n < \omega$. Pick $A_n \subset U_n$ with $|A_n| = \aleph_n$ and set $A = \cup\{A_n : n < \omega\}$. Then $A^\circ = \emptyset$, contradiction.

Now, let $V \supset C$ be open. If we had $|X \setminus V| \geq \aleph_\omega$ then any CAP of $X \setminus V$ would be in C, again a contradiction.

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