Vaught's Conjecture and Boolean Algebras

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Outline

Vaught's Conjecture

Conjecture (Vaught's Conjecture (1961))

If T is a complete first-order theory in a countable language with $n(T) > \aleph_0$, then $n(T) = 2^{\aleph_0}$.

Theorem

If T is

- *a theory of one unary function (Marcus / Miller),*
- *a theory of trees (Steel),*
- *an o-minimal theory (Mayer), or*
- *an ω-stable theory (Shelah, Harrington, and Makkai),*

then $VC(T)$.

Indeed, there are a number of other classes $\mathcal{C}$ of complete first-order theories for which $VC(T)$ is known for all $T \in \mathcal{C}$.

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Vaught's Conjecture for Boolean Algebras

Theorem (Iverson (1991))

If T is a first-order completion of $Th(BA)$, then $n(T) \in \{1, 2^{\aleph_0}\}$.

Proof.

Tarski and Ershov showed the *elementary characteristic*

$$EC(\mathcal{B}) = (p, q, r) \in \{0, 1, \ldots, \omega\} \times \{0, 1, \ldots, \omega\} \times \{0, 1\}$$

of $\mathcal{B}$ characterizes Th($\mathcal{B}$).

Show certain elementary characteristics (p, q, r) have a unique model, namely those of the form $(0, m, 0)$ and $(0, m, 1)$. Show the remaining elementary characteristics (p, q, r) have continuum many models. $\square$

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Elementary Characteristics

Definition

If $\mathcal{B}$ is a Boolean algebra, define its *Ershov-Tarski ideal* to be the set

$$I(\mathcal{B}) := \{x \vee y : x \text{ is atomic and } y \text{ is atomless}\}.$$

Define a sequence $\{\mathcal{B}_i\}_{i\in\omega}$ by $\mathcal{B}_0 := \mathcal{B}$ and $\mathcal{B}_{i+1} := \mathcal{B}_i / I(\mathcal{B}_i)$.

Definition

Define the *elementary characteristic* $EC(\mathcal{B})$ of $\mathcal{B}$ to be the triple

$$\begin{cases} (0,0,0) & \text{if } \mathcal{B} \text{ is trivial} \\ (\omega,0,0) & \text{if } \mathcal{B}_i \text{ is nontrivial for all } i, \\ (p,q,r) & \text{otherwise, where } p \text{ is maximal such that } \mathcal{B}_p \text{ is nontrivial,} \\ & q \leq \omega \text{ is the number of atoms in } \mathcal{B}_p, \text{ and} \\ & r = 1 \text{ if } \mathcal{B}_p \text{ contains atomless elements, else } r = 0. \end{cases}$$

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Borel Completeness

Theorem (Camerlo and Gao (2001))

If T is a first-order completion of $Th(BA)$ with $n(T) = 2^{\aleph_0}$, then the isomorphism problem restricted to models of T is Borel complete.

Proof.

Informally, exhibit the *right* continuum many models having elementary characteristic (p, q, r).

Formally, exhibit a Borel reduction from the isomorphism problem for countable graphs to the isomorphism problem restricted to Boolean algebras with elementary characteristic (p, q, r). □

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Outline

A Review of Infinitary Logic

Definition

The infinitary logic $\mathcal{L}_{\kappa,\lambda}$ allows quantification over fewer than λ many variables, and conjuctions and disjunctions over fewer than κ many subformulas.

Remark

Thus the formulas of $\mathcal{L}_{\omega,\omega}$ are the usual first-order formulas, those with subformulas having finite quantifier depth and finite conjunctions and disjunctions.

Thus the formulas of $\mathcal{L}_{\omega_1,\omega}$ are those with subformulas having finite quantifier depth and *countable* conjunctions and disjunctions.

Vaught's Theorem Rephrased

Theorem (Scott (1965))

If $\mathcal{M}$ is any countable $\mathcal{L}$-structure (with $\mathcal{L}$ countable), there is a sentence $\varphi \in \mathcal{L}_{\omega_1,\omega}$ whose only countable model is $\mathcal{M}$.

Conjecture (Vaught's Conjecture for $\mathcal{L}_{\omega_1,\omega}$)

If φ is a sentence of $\mathcal{L}_{\omega_1,\omega}$ having uncountably many models, then φ has continuum many models.

Question (Camerlo and Gao (2001))

Does $VC(BA)$ hold for the infinitary language $\mathcal{L}_{\omega_1,\omega}$?

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The Theorem (Less a Missing Lemma)

Work In Progress (Kach and Lempp).

$VC(BA)$ for the infinitary language $\mathcal{L}_{\omega_1,\omega}$: If φ is an $\mathcal{L}_{\omega_1,\omega}$ sentence in the language of Boolean algebras extending $Th(BA)$, then φ has continuum many models if it has uncountably many models.

Proof.

Divide into three cases:

- The sentence φ has models of arbitrarily high rank.
- The sentence φ has models of arbitrarily high depth.
- The sentence φ has only models of depth δ and rank ρ.

In each case, exhibit continuum many models of φ.

The Rank Invariant

Definition

If $\mathcal{B}$ is a Boolean algebra, let $I(\mathcal{B})$ be the ideal generated by the atoms of $\mathcal{B}$.

Define a sequence $\{\mathcal{B}_\alpha\}_{\alpha\in\omega_1}$ by $\mathcal{B}_0 := \mathcal{B}$, $\mathcal{B}_{\alpha+1} := \mathcal{B}_\alpha / I(\mathcal{B}_\alpha)$, and $\mathcal{B}_\gamma = \cap_{\beta<\gamma}\mathcal{B}_\beta$.

Definition

The *rank* $\rho(\mathcal{B})$ of a Boolean algebra $\mathcal{B}$ is the least ordinal ρ such that $\mathcal{B}_\rho \cong \mathcal{B}_{\rho+1}$.

Alternately, the rank of $\mathcal{B}$ is the supremum of the ordinals $\beta + 1$ such that $\mathcal{B}$ bounds a β-atom.

Example

$\rho(\mathrm{IntAlg}(1+\eta)) = 0$. $\rho(\mathrm{IntAlg}(2\cdot(1+\eta))) = 1$. $\rho(\mathrm{IntAlg}(\omega^\alpha)) = \alpha + 1$.

Models of Arbitrarily High Rank

Proposition

If φ has models of arbitrarily high rank, then φ has continuum many models.

Proof.

Fix a Δ^0_β formula φ. Fix a model $\mathcal{B} \models \varphi$ containing a $(\beta + 1)$-atom. Show that the Π^0_β theory of $\mathcal{B}$ is unchanged if the $(\beta + 1)$-atom is replaced with a sufficiently *large* Boolean algebra. $\square$

Lemma

Fix an ordinal β. If $\alpha_1, \alpha_2 > \beta$ and σ is a measure with range a subset of $\{\gamma : \gamma > \beta\}$ (i.e., if x is not superatomic, it bounds a β-atom), then

$$Th(IntAlg(\omega^{\alpha_1})) \cap \Pi^0_\beta = Th(IntAlg(\omega^{\alpha_2})) \cap \Pi^0_\beta = Th(\mathcal{B}_\sigma) \cap \Pi^0_\beta$$

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The Depth Invariant

Definition

Define a sequence of sets $\{\Delta^\alpha\sigma(\mathcal{B})\}_{\alpha\in\omega_1}$ by recursion simultaneously for all *uniform* Boolean algebras $\mathcal{B}$, where $\Delta^0\sigma(\mathcal{B}) = \rho(\mathcal{B})$ and

$$\Delta^\alpha\sigma(\mathcal{B}) = \{(\Delta^\beta\sigma(x_1), \dots, \Delta^\beta\sigma(x_n)) : \mathcal{B} = x_1 \oplus \cdots \oplus x_n, \beta < \alpha\}.$$

Definition

The *depth* $\delta(\mathcal{B})$ of a Boolean algebra $\mathcal{B}$ is the least ordinal δ such that $\Delta^\delta\sigma(x) = \Delta^\delta\sigma(y)$ implies $\Delta^{\delta+1}\sigma(x) = \Delta^{\delta+1}\sigma(y)$ for all $x, y \in \mathcal{B}$.

Example

$\delta(\mathrm{IntAlg}(1+\eta)) = 0 = \delta(\mathrm{IntAlg}(2\cdot(1+\eta)))$.
$\delta(\mathrm{IntAlg}((1+\eta)+2\cdot(1+\eta))) = 1$.

Theorem (Ketonen (1978))

The set $\Delta^{\delta(\mathcal{B})+2}\sigma(\mathcal{B})$ is an isomorphism invariant for $\mathcal{B}$.

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Models of Arbitrarily High Depth

Work In Progress (Kach and Lempp).

If φ has models of arbitrarily high depth, then φ has continuum many models.

Proof.

Fix a Δ^0_β formula φ. Fix a model $\mathcal{B} \models \varphi$ of sufficiently large depth. Show that the Π^0_β theory of $\mathcal{B}$ is unchanged if a $(\beta + 1)$-*fishbone* is replaced with a sufficiently *large* Boolean algebra. □

Definition

If $\mathcal{B}$ is the interval algebra of a linear order $\mathcal{L}$, define $\mathcal{B}_\alpha$ to be the Boolean algebra IntAlg$(\mathcal{L} \cdot \omega^\alpha)$.

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Models of Arbitrarily High Depth

Lemma (The Missing Lemma)

Fix an ordinal β. Any Boolean algebra $\mathcal{B}$ with $\delta(\mathcal{B}) \gg \beta$ has a subalgebra (almost) of the form $\hat{\mathcal{B}}_\beta$ for some $\hat{\mathcal{B}}$, with $\Delta^\beta \sigma(\hat{\mathcal{B}}_{\beta_1}) = \Delta^\beta \sigma(\hat{\mathcal{B}}_{\beta_2})$ and $\hat{\mathcal{B}}_{\beta_1} \not\cong \hat{\mathcal{B}}_{\beta_2}$ for all distinct $\beta_1, \beta_2 \geq 0$.

Lemma

Fix an ordinal β. If $\alpha_1, \alpha_2 > \beta$ and σ is a measure with $\mathcal{B}_\gamma$ for $\gamma > \beta$ at coding locations, *then*

$$Th(IntAlg(\mathcal{B}_{\alpha_1})) \cap \Pi^0_\beta = Th(IntAlg(\mathcal{B}_{\alpha_2})) \cap \Pi^0_\beta = Th(\mathcal{B}_\sigma) \cap \Pi^0_\beta$$

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Models of Fixed Depth and Fixed Rank

Proposition

If φ has uncountably many models of depth δ and rank ρ, then φ has continuum many models.

Proof.

Show $\mathbb{B} := \{\mathcal{B} : \mathcal{B} \models \varphi \wedge \chi \wedge \psi\}$ is Borel, where χ states $\delta(\mathcal{B}) = \delta$ and ψ states $\rho(\mathcal{B}) = \rho$.

Let μ be the least ordinal α such that $\{\Delta^\alpha \sigma(x) : x \in \mathcal{B} \in \mathbb{B}\}$ is uncountable. Then $\{\Delta^\alpha \sigma(\mathcal{B}) : \mathcal{B} \in \mathbb{B}\}$ is also uncountable.

Show the latter set is analytic. □

Outline

1. Vaught's Conjecture for First-Order Logic
2. Vaught's Conjecture for the Infinitary Logic $\mathcal{L}_{\omega_1,\omega}$
3. Borel Completeness

Remarks on Borel Completeness

Corollary

If φ has models of arbitrarily high rank, then the isomorphism problem restricted to the models of φ is Borel complete.

If φ has models of arbitrarily high depth, then the isomorphism problem restricted to the models of φ is Borel complete.

If φ has models of rank ρ and depth δ (and no other models), then the isomorphism problem restricted to the models of φ is roughly $\Delta^0_{2\rho+2\delta}$ (thus not Borel complete).

Thanks...

Thanks for your attention!

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