Making classical functions smooth

General Theme

Obtaining smooth (differentiable, or maybe C^n) real-valued functions,
where classical results only produced
functions or maybe continuous functions.

Some papers with my name on them

See my home page.

[1] J. Hart & K. Kunen, Arcs in the Plane,
Topology and Applications, to appear.
[2] K. Kunen, Locally Connected HL Compacta,
Topology and Applications, to appear.
[3] K. Kunen, Forcing and Differentiable Functions
to appear, somewhere ...

See these for proofs, and for references to earlier results.

An Example of the Theme

Two questions: Is it true that
for *all* $B \in [\mathbb{R} \times \mathbb{R}]^{\aleph_1}$,
B can be covered by countably many curves / arcs?

curve = continuous image of $[0, 1]$.
arc = continuous 1-1 image of $[0, 1]$.

very classical: yes for curves (Peano, 1890), you can cover the plane.

somewhat classical: yes for arcs iff
all sets of size $\aleph_1$ are first category.
$\rightarrow$: A countable union of arcs is first category.
$\leftarrow$: First cover B by countably many Cantor sets.
this is possible in $\mathbb{R}$ and hence in $\mathbb{R} \times \mathbb{R}$.
But every Cantor set in the plane is contained in an arc.

So, the situation for arcs is clear under $MA(\aleph_1)$.

Now, focusing on arcs, *Question:*
Can your arcs be C^1 or C^2 or C^3 or $\cdots\cdots$?

Answers:
PFA implies "yes" for C^1.
$MA(\aleph_1)$ is not enough here.
In *ZFC*, "no" for C^2.

See [1][2] for proofs

but to sketch the reason for the "no" for C^2:

Why "no" for C^2:

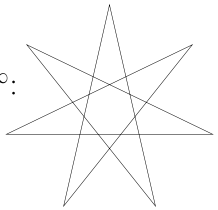
There is a $B \in [\mathbb{R} \times \mathbb{R}]^{\aleph_1}$, such that
B meets every C^2 arc in a finite set. Hence,
you can't cover B with countably many C^2 arcs (or curves).

In fact (Advanced Calculus Exercise), there's actually a
Cantor set $K \subset \mathbb{R} \times \mathbb{R}$ with this property.
Hint: K is very ragged, so that an intersection of K
with a C^2 arc A will contradict Taylor's Theorem
at a limit point of $K \cap A$.

Technical point: for $k \geq 1$,
"C^k arc" means the range of some continuous 1-1 $\Psi : [a, b] \to \mathbb{R} \times \mathbb{R}$
such that Ψ is C^k in the usual sense
and Ψ is *regular*
(Ψ' is never 0; equivalently, the parameter can be arc length)

If you delete "regular" (call this *weakly* C^k), then
under just $MA(\aleph_1)$, every $B \in [\mathbb{R} \times \mathbb{R}]^{\aleph_1}$
can be covered by countably many weakly C^∞ arcs.

Note: This curve is weakly C^∞:



Just slow down at the corners.

Another example of the theme:

On Dense Sets – another example of the theme

$E \subseteq \mathbb{R}$ is $\aleph_1$ – dense iff E meets every open set in a set of size $\aleph_1$.
Cantor: All $\aleph_0$ – dense sets look alike.
What about $\aleph_1$ – dense sets?

More precisely, let $\mathcal{F}$ be the set of all
continuous strictly increasing bijections from $\mathbb{R}$ onto $\mathbb{R}$.

Question (Harvey Friedman, 196?):
Is it consistent that
Whenever $D, E \subseteq \mathbb{R}$ are $\aleph_1$–dense,
there is an $f \in \mathcal{F}$ such that $f(D) = E$?

He knew:
Cantor: Yes for $\aleph_0$–dense. (*and*, f can be C^∞)
No under CH: $\mathbb{R}$ and $\mathbb{R}\backslash\{0\}$ are not homeomorphic.
No in Cohen's model for $\neg CH$:
some are first category and some aren't
Typical 1960s question: OK, what about $MA + \neg CH$?

Baumgartner (1973): it's consistent to have $MA + 2^{\aleph_0} = \aleph_2$ + yes.
In hindsight, his proof shows $PFA \to$ yes.

Avraham and Shelah (1981): Con($MA + 2^{\aleph_0} = \aleph_2$ + no).

New question: Can you actually get $f \in C^n$ ($n \geq 1$)
say, under PFA ?

Some answers:

Answers and More Questions

$E \subseteq \mathbb{R}$ is $\aleph_1$ – dense iff E meets all open intervals in a set of size $\aleph_1$.
$\mathcal{F}$: all continuous strictly increasing bijections from $\mathbb{R}$ onto $\mathbb{R}$.

Theorem 1 (ZFC) There are $\aleph_1$–dense $D, E \subseteq \mathbb{R}$
such that there is no $f \in \mathcal{F} \cap C^1(\mathbb{R})$ with $f(D) = E$.
In fact, there is no $f \in \mathcal{F} \cap C^1(\mathbb{R})$ and $\aleph_1$–dense
$D^* \subseteq D$ and $E^* \subseteq E$ such that $f(D^*) = E^*$.

Question 1 (☹): What about $\mathcal{FD}$?
$\mathcal{FD}$:= the set of functions in $\mathcal{F}$ which are everywhere differentiable.

Theorem 2 (PFA)(partial answer) For any $\aleph_1$–dense $D, E \subseteq \mathbb{R}$
there is an $f \in \mathcal{FD}$ and an $\aleph_1$–dense
$D^* \subseteq D$ such that $f(D^*) = E$.
Question 1 asks: Can you make $D^* = D$?

"Question" 2 (☹): In Theorem 1 (first two lines),
can you (in ZFC) distinguish some D, E by a *quotable* property?
(like, in the Cohen model, D is first category and E isn't)

"Theorem" 3 (ZFC) Yes for C^2.

So, IOU three proofs.

Start with "Theorem" 3, since even the question isn't clear:

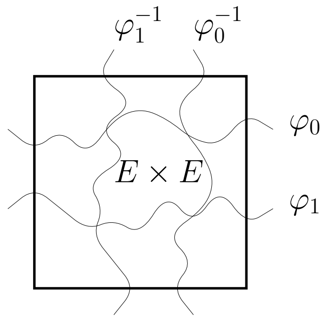
"Proof" of "Theorem" 3 page 1 of 2

"Theorem" 3 (ZFC) There are $\aleph_1$–dense $D, E \subseteq \mathbb{R}$
such that there is no $f \in \mathcal{F} \cap C^2(\mathbb{R})$ such that $f(D) = E$.
Furthermore, there's a "quotable" property involving C^2 stuff
true of D and false of E.
Question 2 was: Can you replace "2" by "1"?

Consider Sierpiński's text: *Hypothèse du Continu*, 1934 & 1956.
Many equivalents of CH, many of which really translate to
ZFC theorems characterizing $\aleph_1$.

Example: Translating his $CH \leftrightarrow P_2$
P_2: "Le plan est une somme
d'une infinité dénombrable de courbes".
you get the ZFC theorem:
$E \times E$ is a countable union of "curves" iff $|E| \leq \aleph_1$.
So, E can be $\mathbb{R}$ under CH.

"curve" in P_2 means: a graph of a function or inverse function:
$|E| \leq \aleph_1 \leftrightarrow$
$\exists \varphi_i : E \to E$ $(i \in \omega)$
s.t. $E \times E = \bigcup_i (\varphi_i \cup \varphi_i^{-1})$
E is just an abstract set;
there's no continuity here.

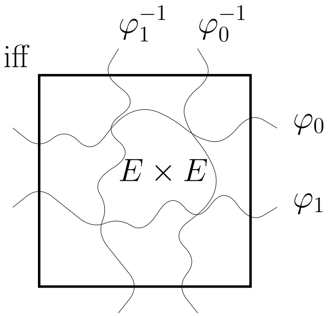


BUT

Question: Suppose $E \subseteq \mathbb{R}$ and $|E| = \aleph_1$.
Can the φ_i be continuous (C^0)? or even smooth ($C^1, C^2, \ldots$)?
The "quotable" property involves the φ_i being C^2.

"Proof" of "Theorem" 3 page 2 of 2

For $n \in \omega \cup \{\infty\}$: $E \subseteq \mathbb{R}$ is n*–small* iff
  $\exists \varphi_i \in C^n(\mathbb{R}, \mathbb{R})$ $(i \in \omega)$
  s.t. $E \times E = \bigcup_i (\varphi_i \cup \varphi_i^{-1})$
  C^0 = "continuous";
  C^∞ means C^n for all $n \in \omega$.



Always, $|E| = \aleph_1$:
  By Sierpiński, $|E| \le \aleph_1$ for every 0–small E, and
  countable sets trivially are ∞–small,

Remarks:
  E is n–small iff $E + \mathbb{Q}$ is n–small.
  $|E| = \aleph_1$ implies $E + \mathbb{Q}$ is $\aleph_1$–dense.
Theorem 3.1 (*ZFC*) There is a D of size $\aleph_1$ which is ∞–small.
Theorem 3.2 (*ZFC*) There is an E of size $\aleph_1$ which is not 2–small.
Then WLOG D, E are $\aleph_1$–dense, and hence "Theorem" 3:
  No C^2 bijection can map D to E.
  "2–small" is the "quotable" property.

More Remarks
  Is every set of size $\aleph_1$ 0–small or even 1–small?

Under *CH*: $\mathbb{R}$ has size $\aleph_1$ and is not 0–small (Baire).

Under $MA(\aleph_1)$: Every E of size $\aleph_1$ is 0–small.

Under *PFA*: Every E of size $\aleph_1$ is 1–small
  so you need a different "quotable" property involving C^1 stuff.

It's consistent to have $MA(\aleph_1)$ plus
  some set of size $\aleph_1$ is not 1–small.

Proof of Theorem 3.2

There is an $E \subseteq \mathbb{R}$ of size $\aleph_1$ which is not 2–small:
  $E^2 \not\subseteq \bigcup_{i \in \omega} (\varphi_i \cup \varphi_i^{-1})$.
    Whenever the $\varphi_i \in C^2(\mathbb{R}, \mathbb{R})$.
That's easy: Fix $B \in [\mathbb{R} \times \mathbb{R}]^{\aleph_1}$, such that
B meets every C^2 arc in a finite set, and let $E = \mathrm{dom}(B) \cup \mathrm{ran}(B)$.

Idea for Theorem 3.1

There's a $D \in [\mathbb{R}]^{\aleph_1}$ which is ∞–small.

Simpler fact: one can get D to be 0–small:
  $D \times D = \bigcup_i (\varphi_i \cup \varphi_i^{-1}) \cup \Delta$, where all φ_i are continuous
  and Δ is the diagonal (identity function).
Proof: First, replace $\mathbb{R}$ by the Cantor set, 2^ω.
Then, some define "nice" $\varphi_i : 2^\omega \to 2^\omega$. Then, choose D.

Let Γ map $\omega \times \omega$ 1-1 into ω. Let $(\varphi_i(x))(j) = x(\Gamma(i, j))$.

"Nice" Lemma: For all countable non-empty $Z \subseteq 2^\omega$,
  there is an $x \in 2^\omega$ such that $Z = \{\varphi_i(x) : i < \omega\}$.
Proof: Let $Z = \{y_i : i \in \omega\}$.
  Let $x(\Gamma(i, j)) = y_i(j)$ for all i, j; then $\varphi_i(x) = y_i$.

Let $D = \{d_\alpha : \alpha < \omega_1\}$ where d_α is chosen recursively
  so that $\{d_\xi : \xi < \alpha\} \subseteq \{\varphi_i(d_\alpha) : i \in \omega\}$
    Then $\xi < \alpha \to (d_\xi, d_\alpha) \subseteq \bigcup_i \varphi_i$
  and so that $d_\alpha \notin \{d_\xi : \xi < \alpha\}$
    so the d_α are all different.
    No problem here — since there's $2^{\aleph_0}$ choices for d_α.

"Proof" of Theorem 1

$\mathcal{F}D$ = the functions in $\mathcal{F}$ which are everywhere differentiable.
Theorem 1 (*ZFC*) There are $\aleph_1$–dense $D, E \subseteq \mathbb{R}$
such that for all $\aleph_1$–dense $D^* \subseteq D$ and $E^* \subseteq E$
and $f \in \mathcal{F}D$ with $f(D^*) = E^*$:
$f'(x) = 0$ for all but countably many $x \in D^*$
so, $f \notin C^1(\mathbb{R})$ because
f' is 0 on a dense set and > 0 on a dense set.

Lemma (advanced calculus exercise).
There are Cantor sets $H, K \subset \mathbb{R}$ such that
$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x_0, x_1 \in H\ \forall y_0, y_1 \in K$
$[0 < |x_1 - x_0| < \delta\ \wedge\ 0 < |y_1 - y_0| < \delta\ \longrightarrow$
$(y_1 - y_0)/(x_1 - x_0) \in (-\varepsilon, \varepsilon) \cup (1/\varepsilon, \infty) \cup (-\infty, -1/\varepsilon)]$

Proof of Theorem 1:
Fix H, K as in the lemma and then fix $\widetilde{H} \in [H]^{\aleph_1}$ and $\widetilde{K} \in [K]^{\aleph_1}$.
Let $D = \bigcup\{\widetilde{H} + s : s \in \mathbb{Q}\}$ and $E = \bigcup\{\widetilde{K} + t : t \in \mathbb{Q}\}$.

Now, suppose we had $f \in \mathcal{F}D$ with $f(D^*) = E^*$,
and $f'(x) > 0$ at $\aleph_1$ points of D^*.
Then there's $\aleph_1$ of these in the same translate, so
some translate contains a convergent sequence of them.
So, we get $d_n \to d_\omega$, all in one $H + s$
and $e_n = f(d_n) \to e_\omega$, all in one $K + t$.
But then $(e_\omega - e_n)/(d_\omega - d_n) \to f'(d_\omega) > 0$,
contradicting the lemma, since translating back to H, K:

$$\frac{e_\omega - e_n}{d_\omega - d_n} = \frac{(e_\omega - t) - (e_n - t)}{(d_\omega - s) - (d_n - s)}\ ,$$

which should get close to 0 or $\pm\infty$ as $n \nearrow \omega$.

"Proof" of Theorem 2

Fix $\aleph_1$–dense $D, E \subseteq \mathbb{R}$.
Assume *PFA*. We produce f, g, D^* such that
1. f is a strictly increasing bijection from $\mathbb{R}$ onto $\mathbb{R}$.
2. $g := f'$ exists everywhere.
3. $D^* \subseteq D$ is $\aleph_1$–dense.
4. $f(D^*) = E$.
5. $g(x) = 0$ for $x \in D^*$.
6. $\{x : g(x) > 0\}$ is dense (obvious from (1)(2)).
7. $\{x : g(x) = 0\}$ is dense (obvious from (5))
   so g is nowhere continuous.

(1)(2)(3)(4) restates Theorem 2.
(1)(2)(6)(7) is a classical *ZFC* construction ($\approx$ 1890).
(5) is to be expected from proof of Theorem 1.

Question 1 was: Can you make $D^* = D$?

Assume *CH* instead of *PFA*, and force f, g, D^* by a ccc poset.
(the "collapse the continuum trick").
CH is needed to make $\mathbb{P}$ ccc.
Amalgamate a classical construction
with Baumgartner's proof (getting a continuous f).
Get continuous $g_n \to g$ and $f_n \to f$ (*pointwise*).
$f_n(x) = \int_0^x g_n(t)\,dt$ and $f(x) = \int_0^x g(t)\,dt$.
A forcing condition gives you some $g_0, \ldots g_n$, $f_0, \ldots f_n$,
and a finite order-preserving $\sigma \subset D \times E$.
The f_n *approximate* σ and converge to $f \supset \sigma$.

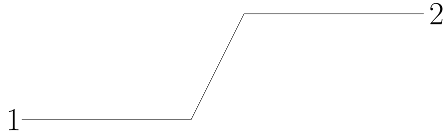
Major problems:
1. Why does f' exist everywhere? (borrow classical ideas)
2. Why is the forcing ccc? (borrow Baumgartner's ideas)

Problem 1

Why does f' exist everywhere?

We have continuous $f_n \to f$ and $g_n \to g$ (*pointwise*).
$\quad f_n(x) = \int_0^x g_n(t)\,dt$.
The g_n are uniformly bounded, so $f(x) = \int_0^x g(t)\,dt$.
The g_n are positive, so f and the f_n are strictly increasing.

The convergence $g_n \to g$ can't be uniform;
$\quad$ you can't get $f \in C^1$, so g won't be continuous.
But you need more to guarantee that $f'(x) = g(x)$ *everywhere*.
Problem: uniform convergence is too much
$\quad$ but pointwise convergence isn't enough:
If $g_n(x)$ is:
1 for $x \le -2^{-n}$
2 for $x \ge 2^{-n}$
linear for $-2^{-n} \le x \le 2^{-n}$



Then $f(x)$ is x for $x \le 0$ and $2x$ for $x \ge 0$, and $f'(0)$ doesn't exist.

So you need to assume a little more about the convergence;
Following Katznelson and Stromberg (Math. Monthly, 1974):
f' will exist and equal g if $g_{n+1} = g_n - \psi_n + \theta_n$, where $\sum_n \theta_n$ converges uniformly and the θ_n, ψ_n are positive functions and:
$\quad \frac{1}{b-a}\int_a^b \psi_n(x)\,dx \le 4\min(\psi_n(a), \psi_n(b)) \quad$ whenever $a < b$.

Remark (why are we doing this?); Following Baumgartner [1973]:
If all you want is a *continuous* $f \in \mathcal{F}$ with $f(D) = E$ then
$\quad$ Each $\sigma \in \mathbb{P}$ is a finite order-preserving bijection; $\sigma \subset D \times E$.
$\quad F_G := \bigcup G : D \to E$ is order-preserving.
$\quad$ Let $f = \mathrm{cl}(F_G) \in \mathcal{F}$, which is continuous (*everywhere*).
$\quad$ The f_n, g_n let you force f to be *differentiable* everywhere.

Problem 2

Why is the forcing ccc?
Start with $\aleph_1$–dense $D, E \subseteq \mathbb{R}$. Assume CH.
$p \in \mathbb{P}$ gives you:
$\quad$ A finite order-preserving $\sigma^p \subset D \times E$.
$\quad$ An $N^p \in \omega$ and g_n^p, f_n^p for $n < N^p$.
$\quad$ In $V[G]$, the f_n will *approximate* σ and converge to $f \supset \sigma$.
$\quad f_n^p(x) = \int_0^x g_n^p(t)\,dt$.

Two obvious uncountable antichains:
1. There's $\aleph_1$ possibilities for $\sigma(d) \in E$.
2. There's $2^{\aleph_0} = \aleph_1$ possibilities for the f_n and g_n.

Fix with elementary submodels:
Let $\langle M_\xi : 0 < \xi < \omega_1\rangle$ be a continuous chain of countable elementary
$\quad$ submodels of $H(\kappa)$, with $D, E \in M_1$. $\quad$ Let $M_0 = \emptyset$.
For $x \in \bigcup_\xi M_\xi$, let $\mathrm{ht}(x)$, the *height* of x,
$\quad$ be the ξ such that $x \in M_{\xi+1} \backslash M_\xi$.
By CH, $\mathrm{ht}(x)$ is defined whenever $x \in \mathbb{R}$ or x is a Borel subset of $\mathbb{R}$.
Avoid antichain 1: For $(d, e) \in \sigma$, $\mathrm{ht}(d)$ and $\mathrm{ht}(e)$ differ finitely.
Avoid antichain 2: Just require that the $g_n^p, f_n^p \in M_1$.

Third obvious uncountable antichain $\{p_\alpha : \alpha < \omega_1\}$, where
$\quad \sigma_\alpha = \sigma^{p_\alpha} = \{(d_\alpha, e_\alpha)\}$ and map $d_\alpha \to e_\alpha$ is order-*reversing*
Fix: Require $\mathrm{ht}(d) > \mathrm{ht}(e)$ for $(d, e) \in \sigma$.

The more standard fix is $\mathrm{ht}(d) \ne \mathrm{ht}(e)$.
$\quad$ But, this is incompatible with the requirements on the f_n, g_n;
$\quad$ But now, the domain of the generic function will be a
$\quad\quad$ *subset* D^* of D.

Hence Question 1: Can you make $D^* = D$?