Self-commuting lattice polynomial functions

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Commuting operations

Let A be an arbitrary set, and n and m positive integers.

We denote $[n] := \{1, \dots, n\}$.

Definition

We say that $f\colon A^n \to A$ and $g\colon A^m \to A$ commute if

$$\begin{aligned} f\bigl(g(a_{11}, a_{12}, \dots, a_{1m}), \dots, g(a_{n1}, a_{n2}, \dots, a_{nm})\bigr) \\ = g\bigl(f(a_{11}, a_{21}, \dots, a_{n1}), \dots, f(a_{1m}, a_{2m}, \dots, a_{nm})\bigr), \end{aligned}$$

for all $a_{ij} \in A$ ($i \in [n]$, $j \in [m]$).

If f and g commute, then we write $f \perp g$.

Commuting operations

In other words, f and g commute if

$$\begin{array}{cccccccc}
 & f(& f(& & f(& & & f(\\
g(& a_{11} & a_{12} & \cdots & a_{1m} &) & = & c_1 \\
g(& a_{21} & a_{22} & \cdots & a_{2m} &) & = & c_2 \\
 & \vdots & \vdots & \ddots & \vdots & & & \vdots \\
g(& a_{n1} & a_{n2} & \cdots & a_{nm} &) & = & c_n \\
 &) &) & &) & & &) \\
 & \parallel & \parallel & & \parallel & & & \parallel \\
g(& d_1 & d_2 & \cdots & d_m &) & = & b
\end{array}$$

A particular case . . .

For $n = m = 2$, we have $f \perp g$ if

$$f\big(g(a_{11}, a_{12}), g(a_{21}, a_{22})\big) = g\big(f(a_{11}, a_{21}), f(a_{12}, a_{22})\big).$$

Theorem (Eckmann–Hilton, 1962)

If f and g are binary operations on A with an identity element and $f \perp g$, then $f = g$ and $(A; f)$ is a commutative monoid.

The relevance of commutation in universal algebra:

Commutation is the defining property of:

1. entropic algebras,
2. modes,
3. centralizer clones,
4. ...

Self-commuting operations

Let A be an arbitrary set, and n a positive integer.

Definition

An operation $f\colon A^n \to A$ is self-commuting (or bisymmetric) if $f \perp f$, that is,

$$\begin{aligned} f\big(f(a_{11}, a_{12}, \ldots, a_{1n}), \ldots, f(a_{n1}, a_{n2}, \ldots, a_{nn})\big) \\ = f\big(f(a_{11}, a_{21}, \ldots, a_{n1}), \ldots, f(a_{1n}, a_{2n}, \ldots, a_{nn})\big), \end{aligned}$$

for every $a_{ij} \in A$.

A particular case . . .

An algebra $(A; f)$ where f is a binary operation that satisfies the identity

$$f\big(f(a_{11}, a_{12}), f(a_{21}, a_{22})\big) = f\big(f(a_{11}, a_{21}), f(a_{12}, a_{22})\big)$$

is called a medial groupoid.

Thus, the notion of self-commutation generalizes mediality.

Lattice polynomial functions

Let $(L; \wedge, \vee)$ be a lattice with least and greatest elements 0 and 1, respectively.

Definition

A (lattice) polynomial function is any map $p : L^n \to L$ which is a composition of

1. the lattice operations $\wedge$, $\vee$,
2. projections $\mathbf{x} \mapsto x_i$, $i \in [n]$, and
3. constant functions $\mathbf{x} \mapsto c$, $c \in L$.

Representations: disjunctive normal form

A function $p\colon L^n \to L$ has a disjunctive normal form (**DNF**) if

$$p(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(a_I \wedge \bigwedge_{i \in I} x_i\Big)$$

for some $a_I \in L$ ($I \subseteq [n]$).

Representations: disjunctive normal form

Proposition (Goodstein 1965)

Let $(L; \wedge, \vee)$ be a bounded distributive lattice. A function $p\colon L^n \to L$ is a polynomial function if and only if it has the **DNF**

$$p(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(p(\mathbf{e}_I) \wedge \bigwedge_{i \in I} x_i \Big),$$

where for $I \subseteq [n]$, $\mathbf{e}_I \in \{0,1\}^n$ is the characteristic vector of I:

$$(\mathbf{e}_I)_i = \begin{cases} 1 & \text{if } i \in I, \\ 0 & \text{if } i \notin I. \end{cases}$$

A few consequences ...

Corollary

Let L be a bounded distributive lattice. Every polynomial function $p\colon L^n \to L$ is uniquely determined by its restriction to $\{0,1\}^n$.

Corollary

Every polynomial function $p\colon L^n \to L$ over a bounded distributive lattice L has a **DNF**

$$p(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(a_I \wedge \bigwedge_{i \in I} x_i\Big),$$

where $a_I \leq a_J$ whenever $I \subseteq J$.

Our problem

Problem

Explicitly describe the self-commuting lattice polynomial functions.

Sufficient conditions: weighted disjunction

A polynomial function $f\colon L^n \to L$ is a weighted disjunction if

$$f(x_1, x_2, \ldots, x_n) = a_\emptyset \vee \bigvee_{i\in[n]} (a_i \wedge x_i)$$

for some $a_\emptyset, a_1, \ldots, a_n \in L$.

Lemma

Let L be a distributive lattice. If $f\colon L^n \to L$ is a weighted disjunction, then it is self-commuting.

Sufficient conditions: chain form

We say that $f\colon L^n \to L$ has chain form if

$$f(x_1, x_2, \ldots, x_n) = a_\emptyset \vee \bigvee_{i \in [n]} (a_i \wedge x_i) \vee \bigvee_{1 \leq \ell \leq r} \Big(a_{S_\ell} \wedge \bigwedge_{i \in S_\ell} x_i\Big),$$

where $r \geq 1,\ |S_1| \geq 2,$ and

1. $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_r \subseteq [n]$, and
2. for all $i \in [n]$, there is a $j \in S_1$ such that $a_i \leq a_j$.

Sufficient conditions: chain form

Lemma

Let L be a distributive lattice. If $f\colon L^n \to L$ has chain form, then it is self-commuting.

Corollary

Every binary polynomial function over a distributive lattice is self-commuting.

Example 1

Consider $f\colon [0,1]^3 \to [0,1]$ given by $\quad f = (x_1 \wedge x_2) \vee (x_2 \wedge x_3)$.

$$\begin{array}{ccccccc}
 & f(& f(& f(& & & f(\\
f(& 0 & 1 & 1 &) = & & 1 \\
f(& 1 & 1 & 0 &) = & & 1 \\
f(& 0 & 0 & 0 &) = & & 0 \\
 &) &) &) & & &) \\
 & \parallel & \parallel & \parallel & & & \parallel \\
f(& 0 & 1 & 0 &) = & \color{red}{0 \neq} & \color{red}{1}
\end{array}$$

Thus f is not self-commuting!

Necessary conditions: chain form

We say that $f\colon L^n \to L$ has chain form if

$$f(x_1, x_2, \ldots, x_n) = a_\emptyset \vee \bigvee_{i\in[n]} (a_i \wedge x_i) \vee \bigvee_{1\leq\ell\leq r} \Big(a_{S_\ell} \wedge \bigwedge_{i\in S_\ell} x_i\Big),$$

where $r \geq 1$, $|S_1| \geq 2$, and

1. $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_r \subseteq [n]$, and
2. for all $i \in [n]$, there is a $j \in S_1$ such that $a_i \leq a_j$.

Example 2

Consider $f\colon [0,1]^3 \to [0,1]$ given by $\quad f = (0.5 \wedge x_1) \vee (x_2 \wedge x_3)$.

$$
\begin{array}{cccccccc}
 & f(& f(& f(& & & & f(\\
f(& 0 & 0 & 1 &) = & & & 0 \\
f(& 0 & 1 & 0 &) = & & & 0 \\
f(& 0 & 1 & 1 &) = & & & 1 \\
 &) &) &) & & & &) \\
 & \parallel & \parallel & \parallel & & & & \parallel \\
f(& 0 & 1 & 0.5 &) = & {\color{red}0.5} & {\color{red}\neq} & {\color{red}0}
\end{array}
$$

Thus f is not self-commuting!

Description of bisymmetric polynomial functions on chains

Theorem (Couceiro, Lehtonen 2010)

Let $(L; \wedge, \vee)$ be a bounded chain. A polynomial function $f\colon L^n \to L$ is self-commuting if and only if

1. it is a weighted disjunction, or
2. it has chain form.

Open problems

1. Determine whether these conditions are still necessary in the general case of distributive lattices.

2. Find necessary and sufficient conditions for two lattice polynomial functions to commute.

3. Characterize “strongly bisymmetric” lattice polynomial functions, i.e., functions

$$f\colon \bigcup_{n\geq 1} L^n \to L$$

such that for all $n \geq 1$, $f_n := f|_{L^n}$ is a lattice polynomial function and for all $n, m \geq 1$, $f_n \perp f_m$.

Thank you for your attention!