A survey on the Katětov order

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Definition

The *Katětov order* is defined as follows: Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. Then $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f \in \omega^\omega$ such that $f^{-1}(I) \in \mathcal{J}$ for all $I \in \mathcal{I}$.

Definition

We say an ideal $\mathcal{I}$ is *tall* if for every $X \in [\omega]^\omega$, $[X]^\omega \cap \mathcal{I} \neq \emptyset$.

Tall and non-tall ideals

The following are equivalent:

- $\mathcal{I}$ is tall,
- $\mathcal{I} \not\leq_K \mathbf{Fin}$, and
- $\mathcal{I} \restriction X \neq \mathbf{Fin}(X)$ for all $X \in \mathcal{I}^+$.

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Some order theoretic properties

Global order-theoretic properties

- Prime ideals are cofinal and MAD families are coinitial.
- Every family of at most $\mathfrak{c}$ ideals has a $\leq_K$-lower bound.
- Below any tall ideal $\mathcal{I}$, there is a $\leq_K$-antichain of size $\mathfrak{c}$.
- Below any tall ideal $\mathcal{I}$, there is a decreasing chain of length $\mathfrak{c}^+$.

Order theoretic properties among definable ideals

- There is an order-embedding from $\mathcal{P}(\omega)/\mathbf{Fin}$ into the family of sumable ideals.
- The family of the Cantor-Bendixson's ideals is an increasing chain of Borel ideals of length ω_1.

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Let E be a random graph on ω. $\mathcal{R}$ is the ideal generated by the homogeneous sets of E.

Theorem

$\omega \to (\mathcal{I}^+)^2_2$ if and only if $\mathcal{R} \not\leq_K \mathcal{I}$.

E. Thümmel provided an example of a Borel ideal $\mathcal{I}$ which satisfies the arrow property. Unfortunately this example is non-homogeneous. Actually Thümmel's example has a positive set X such that the restriction $\mathcal{I} \restriction X$ does not satisfy $X \to (\mathcal{I}^+)^2_2$.

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Definition

Let $\varphi : [\omega]^n \to \omega$. A subset A of ω is $\mathcal{I}$-*homogeneous* for φ if $\varphi''[A]^n \in \mathcal{I}$.

Definition

$\mathcal{I}$ satisfies the property $\omega \to (\omega)^n_{\omega,\mathcal{I}}$ if for every $\varphi : [\omega]^n \to \omega$ there is an infinite $\mathcal{I}$-homogeneous set.

For each n, there is a critical ideal for the property $\omega \to (\omega)^n_{\omega,\mathcal{I}}$ in the Katětov order, the ideal $\mathcal{G}^n_c$ which is generated by the subsets A of $[\omega]^n$ so that $[X]^n \subseteq A$ implies X is finite. This is:

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With respect to the property $\omega \to (\omega)^2_{\omega,\mathcal{I}}$ we have that:

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- It is not the case that $\omega \to (\omega)^2_{\omega,\mathbf{CB2}}$, and
- $\omega \to (\omega)^2_{\omega,\mathbf{CB3}}$.
- In general, $\omega \to (\omega)^n_{\omega,\mathbf{CB}(n+1)}$ for all n and then,
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A cardinal invariant related with Katětov order

$$\mathrm{cov}^*\mathcal{I} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall X \in [\omega]^\omega)(\exists A \in \mathcal{A})|X \cap A| = \aleph_0\}.$$

The link

If $\mathcal{I} \leq_K \mathcal{J}$ then $\mathrm{cov}^*(\mathcal{J}) \leq \mathrm{cov}^*(\mathcal{I})$.

Theorem

If $\mathcal{I}$ is a Borel ideal and $\mathrm{cov}^*(\mathcal{I}) \leq \mathrm{cov}(\mathcal{M})$ then $\omega \to (\omega)^2_{\omega,\mathcal{I}}$.

Sketch of proof.

1. $\mathfrak{par}_2 \leq \mathrm{cov}^*(\mathcal{G}^2_c)$.
2. (Blass) $\mathfrak{par}_2 = \min\{\mathfrak{b}, \mathfrak{s}\} \geq \mathfrak{h}$.
3. In the Mathias model, $\mathrm{cov}^*(\mathcal{I}) \leq \mathrm{cov}(\mathcal{M}) < \mathfrak{h} \leq \mathrm{cov}^*(\mathcal{G}^2_c)$.
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Let $\mathcal{I}$ be a Borel ideal. Then, either $\mathcal{I} \leq_K$ **nwd** or there is $X \in \mathcal{I}^+$ such that $\mathcal{ED} \leq_K \mathcal{I} \restriction X$.

Sketch of proof.

- Let's play a game G:

Player I	$I_0 \in \mathcal{I}$		$I_1 \in \mathcal{I}$		$\cdots$
Player II		$n_0 \in \omega \setminus I_0$		$n_1 \in \omega \setminus I_1$	$\cdots$

  Player I wins if $\{n_k : k < \omega\} \in \mathcal{I}$.

- **Claim.** Player I has a winning strategy for G iff there is an $\mathcal{I}^*$-branching tree of increasing sequences st $rng(x) \in \mathcal{I}$ for all branch x of T.

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Player I	$I_0 \in \mathcal{I}$		$I_1 \in \mathcal{I}$		$\cdots$
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Player I wins if $\{n_k : k < \omega\} \in \mathcal{I}$.

- **Claim.** Player I has a winning strategy for G iff there is an $\mathcal{I}^*$-branching tree of increasing sequences st $rng(x) \in \mathcal{I}$ for all branch x of T.

Hrušák's Category Dichotomy

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Let $\mathcal{I}$ be a Borel ideal. Then, either $\mathcal{I} \leq_K$ **nwd** or there is $X \in \mathcal{I}^+$ such that $\mathcal{ED} \leq_K \mathcal{I} \restriction X$.

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