Topological algebra based on sorts and properties as free and cofree universes

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MOTIVATION

A well-structured category $\mathcal{C}$ should satisfy WS1-5.

WS1. The objects $U, V, \ldots$ of $\mathcal{C}$ are organized as universes equipped with algebraic or spatial structure and topological or localic structure listing possible paintings or (abstract) worlds.

WS2. Algebra is organized in terms of elements or points of universes classified by sort $s, t, \ldots$. Topology is organized in terms of states or opens in universes each interpreting some property p of the elements. We generalize topology to permit more than one property.

WS3. Universal algebra is furnished with operations $f : s \to t$ between sorts as functions acting on elements. Universal topology is furnished with dependencies $d : p \to q$ between properties as functions on states, as yet another generalization.

WS4. Operations and dependencies are governed by equations.

WS5. $\exists$ enough free and cofree universes to contain all terms.

EXAMPLES

Grph 2 sorts: *vertex*, *edge*. $s, t : E \to V$. Discrete (no properties).

Grp ω sorts: G^i, $i < \omega$. $e : G^0 \to G^1$, $m : G^2 \to G^1$. Discrete.

Set 1 sort: *element*. Discrete.

Top $s = \{1\}$, $p =$ Sierpinski sp., 3 dependencies.

Vct$_k$ ω sorts and properties $s^n = p^n = k^n$. Ops & deps l.t.'s

Chu(**Set**, K) As for **Top** $s = \{1\}$, $p = K$, both rigid.

Ab$_{LC}$ (locally compact abelian groups) As for **Top**. $K = \mathbb{R}/\mathbb{Z}$.

κ-*Locales* κ-$\mathrm{Frm}^{\mathrm{op}}$: No sorts (pointless), κ properties K^α $\alpha < \kappa$.
$\bigvee_\alpha : L^\alpha \to L$ for $\alpha < \kappa$, $\wedge_n : L^n \to L^1$ for $n < \omega$.

Stone spaces Locales with κ and Frm replaced by ω and Bool.

Simple ontology of domestic pets: 3 sorts: *cat*, *dog*, *mammal*
2 operations: $is_{cm} : cat \to mammal$, $is_{dm} : dog \to mammal$
3 properties: *weight*, *color*, *hue*
1 dependency: *hue-of*: $color \to hue$.

OUTLINE

1. PROGRAM: topoalgebra = Yoneda + duality. Simple, general.
2. PRETAC: Pretopoalgebraic category $\mathcal{C} = (C, S, P)$
   Algebra $S \subseteq \mathrm{ob}(C)$ consists of sorts $s, t, \ldots$
   Topology $P \subseteq \mathrm{ob}(C)$ consists of properties (attributes) $p, q, \ldots$
3. DEFINITIONS:
   element (point) $a : s \to U$, forming carrier $U_s = \mathrm{Hom}(s, U)$, $s \in S$
   state (open) $x : U \to p$ forming cocarrier $U^p = \mathrm{Hom}(U, p)$, $p \in P$
   map $h : U \to V$, acts on left on elements and on right on states
   operation $f^* : t \to s$, dependency $d : p \to q$
   qualia $k : s \to p$ form a field K_s^p. Inner product $U_s \times U^p \to K_s^p$
   universe U, free univ. $F_s = s$, cofree univ. $K^p = p$. $F_s^p = K_s^p$
4. THEOREM: Every map is homomorphic and continuous
5. TAC = complete dense extension of a pretac.
6. ARITY = functorial sorts (s^2) & properties (algebra, coalgebra,...)

1a. PROGRAM: Algebra

Translated into algebraic language, the Yoneda lemma treats multi-sorted unary algebras.

That is, an algebra $(U_s, U_t, \ldots, f, g, \ldots)$ consists of a family $\langle U_s \rangle_{s \in S}$ of sets indexed by sorts $s \in S$ and a list of unary operations $f : U_s \to U_t$ between those sets.

Up to isomorphism there is one free algebra generated by a variable $\mathbf{x}_s$ for each sort s; let F_s denote a specific such.

Yoneda: $\exists$ bijections $\alpha_s : U_s \to \mathrm{Hom}(F_s, U)$ and $\beta_{st} : \mathrm{Clo}_{st}(U) \to \mathrm{Hom}(F_t, F_s)$ s.t. for all sorts $s \in S$, for all $a \in U_s$ and all $f_U : U_s \to U_t$ in the clone $\mathrm{Clo}_{st}(U)$, $\alpha_t(f_U(a)) = \alpha_s(a)\beta_{st}(f_U)$.

That is, the applicative structure internal to U is dually imitated by the homomorphic structure between U and the free algebras on one generator.

In this view n-ary operations $U^n \to U$ are mimicked as homorphisms from F_s to F_{s^n} where s^n is a product sort, treated later.

1b. PROGRAM: Topology

Ordinary topology can be understood as dual to algebra, with a single dual sort or property *in* giving rise to a cofree algebra K^{in}, namely the Sierpinski space. The two points of K^{in}, as the two continuous functions from 1 to K^{in}, are to be understood as the two possible values of the *in* property, making this property a predicate.

An open x of a space S is a morphism from S to K^{in}. A point $a : 1 \longrightarrow S$ is *in* open x just when $xa : 1 \longrightarrow K^{in}$ is the point 1 of K^{in}.

Our more general notion of topology provides for a set P of properties, as well as for dependencies between properties dual to the operations between sorts.

2. PRETAC

A pretopoalgebraic category $\mathcal{C} = (C, S, P)$, or pretac for short, is a bipointed category, meaning an (abstract) category C with distinguished objects of two kinds forming respective sets S and P.

Every category becomes a pretac by specifying S and P.

This selection imbues all objects of the category with concrete topoalgebraic structure, including the selected objects, and makes all morphisms concrete maps acting on that structure.

Different selections turn the same object into different universes and the same morphism into different maps.

We will show that every map is homomorphic and continuous with respect to the imputed topoalgebraic structure.

The "pre" indicates that not all possible universes need exist in a pretac, even up to isomorphism. It furthermore indicates that the continuous homomorphisms between two universes that are present need not appear exactly once: they may be absent, or be duplicated.

3a. DEFINITIONS: Universes and maps

A choice of sorts and properties elevates each object to the status of universe.

A universe is an object of C in the context of a choice of sorts and properties.

A universe is free (cofree) when it is isomorphic in C to a sort (property).

Accordingly we denote sort s by F_s and dually property p by K^p when treating them as universes.

Universes isomorphic to a sort or property are themselves free or cofree, but will not in general be the canonical such.

A map is a morphism of C in the context of a choice of sorts and properties.

3b. DEFINITIONS: Elements and terms

An element of a universe U of sort s is a map $a : F_s \to U$. We denote the set of elements of U of sort s by $U_s = \mathrm{Hom}(s, U)$.

A term of sort $s \to t$ is an element of F_s of sort t, that is, a map $f^* : F_t \to F_s$. Each universe U interprets each such term f^* contravariantly as the function $f_U : U_s \to U_t$ mapping each element $a \in U_s$ (that is, $a : F_s \to U$) to $af : F_t \to U$, i.e. $af^* \in U_t$.

The notation f^* indicates that the term is to be interpreted contravariantly, namely by right action on elements a. The right action can be understood as substitution of the term f in a.

3c. DEFINITIONS: States, dependencies, qualia

A state of a universe U expressing property p is a map $x : U \to K^p$. We denote the set of states of U expressing property p by $U^p = \mathrm{Hom}(U, p)$.

A dependency of property q on property p is a state of p expressing q, that is, a morphism $d : K^p \to K^q$. Each universe U interprets each such d as the function $d_U : U^p \to U^q$ mapping each state $x \in U^p$, as $x : U \to K^p$, to $dx : U \to K^q$, i.e. $dx \in U^q$.

Example: The Sierpinski space K^{in} has three dependencies on itself, namely its three opens.

Qualia are schizophrenically states of free universes and elements of cofree universes. In particular the sets F_s^p and K_s^p are equal. K^{in} has two elements, of sort *point*.

3d. DEFINITIONS: Maps

For each morphism $h : U \to V$,

1. the left action of h on U_s is the function $h_s = \mathrm{Hom}(s, h) : U_s \to V_s$ defined by $h_s(a) = ha$ for all $a \in U_s$.
2. the right action of h on V^p is the function $h^p = \mathrm{Hom}(h, p) : V^p \to U^p$ defined by $h^p(x) = xh$ for all $x \in V^p$.
3. the action of h is the pair (h_S, h^P) consisting of the family h_S of left actions $h_s : U_s \to V_s$ and the family h^P of right actions $h^p : V^p \to U^p$.

A map is a morphism interpreted by S and P as an action.

Different choices of S and P interpret the same morphism as different maps.

4. Continuous and homomorphic

A map $h : U \to V$ is is homomorphic when its left action on elements of U commutes with all operations f_U of U.

It is continuous when its inverse maps states of V to states of U and commutes with all dependencies d^U of U. Here the inverse image under $h : U \to V$ of a state x of V is the state xh of U.

Theorem *Every map is homomorphic and continuous.*

Proof The left action of $h : U \to V$ on $f_U(a)$ realized as af is $h(af)$. By associativity this is $(ha)f$ or $f_U(h(a))$. States are trivially preserved by inverse image, and maps commute with dependencies by associativity, dual to the argument for operations. □

5a. Equivalence

The equational theory of a pretac consists of the commutative diagrams formed exclusively from operations, dependencies, and qualia.

It is immediate that for every universe U, the operations and dependencies as interpreted in U satisfy the equations of the theory.

Maps $h, k : U \rightarrow V$ are called equivalent when they have the same action.

Proposition *Equivalence is identity for elements and states.*

Proof Let a, b be equivalent elements of U_s. Then $a = a1_s = b1_s = b$. For equivalent states x, y of U^p, $x = 1_p x = 1_p y = y$. □

This applies *a fortiori* to operations and dependencies, being respectively elements and states.

5b. Actions are adjoint

Call an action adjoint when for every left action h_s and right action h^p we have $xh_s(a) = h^p(x)a$.

Proposition The action of every map is adjoint.

Proof $h_s(a)$ is given by ha and $h^p(x)$ by xh. Hence adjointness holds just when $x(ha) = (xh)a$. But this is just associativity. $\square$

Call a pretac extensional when equivalence is identity for all maps.

Proposition *Equivalence is a congruence.*

Corollary *The quotient of a pretac by equivalence is an extensional pretac.*

We assume henceforth that all pretacs are extensional.

5c. Dense extensions

A pretac (C', S, P) extends (C, S, P) when C is a subcategory of C' and all elements and states in C' of objects of C are in C. That is, extension adds no new elements or states to universes of C.

In particular extension adds no new operations, dependencies, or qualia.

A pretac is dense (in the sense of densely embedding S and P) when every proper (nonidentity) extension has a new object.

5d. Dense extensions

Proposition Every pretac has a dense extension with no new objects.

Proof For each non-sort U, non-property V, and adjoint action $U \to V$ not representing a map $h : U \to V$, adjoin $h : U \to V$ with composites with elements of U and states of V defined to correspond to that action. Adjointness ensures associativity of yha for all elements a of U and states y of V. Define each remaining composite $hg : T \to V$ for non-elements $g : T \to U$ to be the map represented by the composition of the actions of h and g, and likewise for composites $kh : U \to W$ for non-states $k : V \to W$. $\square$

5e. Complete pretacs

A dense pretac is complete when it is equivalent to its every dense extension.

A tac (for TopoAlgebraic Category) is a complete dense pretac.

Density of the rationals in any Archimedean field, and the reals as a complete Archimedean field, are posetal instances of these notions.

6a. ARITY: Secondary sorts and properties

To accommodate algebra and coalgebra with other operations than those between two sorts, such as $f : s^2 \to s$ and $f : s \to s + s$, we introduce the notions of secondary sort and property, for example s^2, $s + s$, p^2, $p + q$, etc.

A secondary sort may participate in the signature as either the domain or codomain of operations, dependencies, and qualia.

Examples. (note contravariance)

1. Group multiplication $m^* : s \to s^2$
2. Projections $\pi_1^*, \pi_2^* : s \to s^2$
3. Coalgebra $d^* : A + A \to A$

6b. ARITY: Projections and diagonal

The projections $\pi_1^*, \ldots, \pi_n^* : s \to s^n$ interpret morphisms $t : s^n \to U$ as n-tuples $(t\pi_1, \ldots, t\pi_n)$ where each $t\pi_i \in U_s$.

Inclusions $i^* : s + t \to s$, $j^* : s + t \to t$ work dually, viz.
$\forall a \in U_s.\exists! ai^* \in U_{s+t}$, and
$\forall b \in U_t.\exists! aj^* \in U_{s+t}$.

6c. ARITY: Limits on secondary sorts and properties

In a nonextensional pretac each such n-tuple of elements of sort s in a universe U may appear any number of times in $\mathrm{Hom}(s^n, U)$.

In a dense pretac we want each such n-tuple to appear exactly once.

This can be accomplished neatly by treating secondary sorts and properties as ordinary universes inasmuch as they do not participate in the definitions of element, state, action, extensionality, and extension, and moreover compose only with their associated projections or inclusions, at first.

Extensionality then identifies multiple appearances of an n-tuple, while density creates all n-tuples.

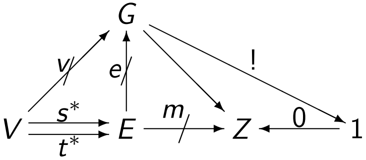
$\mathrm{Hom}(s^n, U)$ is then in a natural bijection with U_s^n.

This approach dualizes to coproducts, and more generally to limits and colimits.

The remaining composites are then filled in when possible. When not the offending object is discarded (full subcategory).

6d. Example

Take the two sorts to be V and E as usual for graphs. Take the one primary property to be Z as the bi-infinite path (a connected acyclic directed graph, unit in-degree and out-degree for all vertices), Take 1 to be the 0-ary property p^0, aka a final object.



where $mt^* = ms^* + 1$. The map $0!$ forces a state labeling edge $e \in G_V$ with 0, for every $v \in G_V$.

Graphs labelable in this way are just the regular graphs, namely those for which every path from vertex u to vertex v has the same length.

Call a graph discrete when there exist enough states from G to Z to label any given vertex or edge with all possible integers.

The discrete graphs in this tac form the category of regular graphs.