Geodesic spaces : momentum
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Groups : symmetry

Vaughan Pratt
Stanford University

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1. Reprise of the relevant bits of my BLAST ’08/9 talks

Goal: express each of the five postulates of Book I of Euclid’s *Elements* equationally.

Their bilinear content is confined to the 3rd and 4th postulates, concerning respectively circles and right angles.

Bilinearity is equational. ✓

But those equations depend on numbers, which Book I outlawed. The obvious trick of identifying the Euclidean line with the underlying field would appear to inevitably lose information in a way that prevents the square of the line from being a Euclidean space.

Absent bilinearity we have only affine spaces.

*Question*: Can each of postulates 1, 2, and 5 be written equationally, observing the proscription on numbers, so that together they define a variety of affine spaces over some field $k$?

*Answer*: Yes, for $k$ each of $\mathbb{Q}$ and $\mathbb{Q}[i]$ (complex rationals)
2. Approach

1. We defined a variety $\text{Grv}$ of "groves" with a binary operation $ab$ denoting the point to which segment $AB$ must be produced to double its length, interpretable in $\text{Ab}$ as $ab = 2b - a$. Writing $abc$ for $(ab)c$, we expressed Postulate 2 as $aa = abb = a$, while Postulate 5 became $ab(cd) = ac(bd)$.

2. We equipped $\text{Grv}$ with $\omega$ many commutative but non-associative $n$-ary centroid operations $a_1 \oplus a_2 \oplus \ldots \oplus a_n$. We wrote Postulate 1 as two equations

$$a_1 \oplus \ldots \oplus a_{n-1} \oplus ((a_1 \oplus \ldots \oplus a_{n-1}) \rightarrow^n b) = b$$
$$((a_1 \oplus \ldots \oplus a_{n-1}) \rightarrow^n (a_1 \oplus \ldots \oplus a_n) = a_n,$$

for each centroid operation in terms of $ab$ ($a \rightarrow^4 b = abab$ etc.)

We showed that the resulting variety is equivalent to $\text{Aff}_\mathbb{Q}$.

3. We extended $\mathbb{Q}$ to $\mathbb{Q}[i]$ with a binary operation $a \cdot b$ denoting $b$ rotated 90 degrees about $a$.

End of reprise.
3. This talk; Geodesic spaces

At FMCS (Vancouver May 2009) Pieter Hofstra asked:
Can non-Euclidean geometry be treated analogously?
My answer (weeks later): weaken Postulate 5 to right distributivity,

\[ abc = ac(bc). \]

Thinking of \( ba, a, b, ab \), etc. as points evenly spaced along a geodesic \( \gamma \), right distributivity expresses a symmetry of \( \gamma \) about an arbitrary point \( c \), namely that the inversion \( \gamma c \) in \( c = \ldots, bac, ac, bc, abc, \ldots \) is itself a geodesic, namely \( \ldots, bc(ac), ac, bc, ac(bc), \ldots \).

These algebras have sometimes been identified with quandles as used to algebraicize knot theory. This is wrong because the quandle operations interpreted in \( \text{Grp} \) are \( b^{-1}ab \) and \( bab^{-1} \), which collapse in \( \text{Ab} \) to \( ab = a \), whereas the above is \( ba^{-1}b \) which is very useful in \( \text{Ab} \).
4. Geodesic theory

A geodesic space or **geode** is an algebraic structure with a binary operation \( x \rightarrow y \), or \( xy \), of **extension** (with \( xyz \) for \((xy)z\)) satisfying

\[
\begin{align*}
&G0 \quad xx = x \\
&G1 \quad xyy = x \\
&G2 \quad xyz = xz(yz)
\end{align*}
\]

Geometrically, segment \( A_0A_1 \) is *extended to \( A_2 = A_0 \rightarrow A_1 \) by producing \( A_0A_1 \) to twice its length: \( |A_0A_2| = 2|A_0A_1| \).

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**Examples**

*Symmetric spaces*: Affine, hyperbolic, elliptic, etc.

*Groups*: Interpret \( x \rightarrow y \) as \( yx^{-1}y \) (abelian groups: \( 2y - x \))

*Number systems*: Integers, rationals, reals, complex numbers, etc.

*Combinatorial structures*: sets, dice, etc.
5. Geodesics

A **discrete geodesic** $\gamma(A_0, A_1)$ is a subspace generated by $A_0, A_1$.

A **geodesic** in $S$ is a directed union of discrete geodesics in $S$.

**Examples:** $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $E$ (§11). Not $\mathbb{R}$ (not fully represented).

Geodesics properly generalize cyclic groups.

**Example:** $E = \mathbb{Z}_4/\{0 = 2\}$. $\bullet \overset{1}{\longrightarrow} \overset{2}{\longrightarrow} 0 \overset{3}{\longrightarrow} \bullet$

$S$ is **torsion-free** when every finite geodesic in $S$ is a point.

The **connected components** of $\gamma(A_0, A_1)$ are $\ldots, A_{-2}, A_0, A_2, \ldots$ and $\ldots A_{-1}, A_1, A_3, \ldots$. These become one component just when $A_0 = A_{2n+1}$ for some $n$, as with $\mathbb{Z}_3$, $\mathbb{Z}_5$, etc.

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**The category $Gsp$**

**Geode homomorphism:** a map $h : S \to T$ s.t. $h(xy) = h(x)h(y)$.

Denote by $Gsp$ the category of geodes and their homomorphisms.
6. Sets

Theorem 1. For any space $S$, the following are equivalent.

(i) $\gamma(A, B) = \{A, B\}$ for all $A, B \in S$ (cf. $\gamma(N, S)$, N&S poles).
(ii) The connected components of $S$ are its points.
(iii) $xy = x$ for all $x, y \in S$.

A set is a geode $S$ with any (hence all) of those properties.

Define $U_{\text{Set}Gsp} : \text{Set} \to \text{Gsp}$ as $U_{\text{Set}Gsp}(X) = (X, \pi_{1}^{2})$, i.e. $xy \overset{\text{def}}{=} x$.

Left adjoint $F_{\text{Gsp}Set}(S)$ = the set of connected components of $S$.

Cf. $D : \text{Set} \to \text{Top}$ where $D(X) = (X, 2^{X})$, a discrete space.

These embed $\text{Set}$ fully in $\text{Top}$ (Pos, Grph, Cat, etc.) and $\text{Gsp}$.

In $\text{Top}$ etc. the embedding $D$ preserves colimits.

In $\text{Gsp}$ the (reflective) embedding $U_{\text{Set}Gsp}$ preserves limits!

In $\text{Set}$, $1 + 1 = 2$ and $2^{\aleph_{0}} = \beth_{1}$ (discrete continuum).
In $\text{Top}$, $1 + 1 = 2$ but $2^{\aleph_{0}} = $ Cantor space, not discrete.
In $\text{Gsp}$, $2^{\aleph_{0}} = \beth_{1}$, discrete (!), but $1 + 1 = \mathbb{Z}$, a homogeneous (no origin) geodesic with two connected components.
7. Normal form terms and free spaces

A normal form geodesic algebra term over a set $X$ of variables is one with no parentheses or stuttering, namely a finite nonempty word $x_1x_2\ldots x_n$ over alphabet $X$ with no consecutive repetitions.

**Theorem 2.** All terms are reducible to normal form using G0-G2. (G2 removes parentheses while G1 and G0 remove repetitions.)

**Theorem 3.** The normal form terms over $X$ form a geode.

Denote this space by $F(X)$, the free space on $X$ consisting of the “$X$-ary” operations. $F(\{\}) = 0$ (initial), $F(\{0\}) = 1$ (final).

$F(\{0, 1\}) = 1 + 1$ has two connected components $0\alpha$ and $1\alpha$.

It is an infinite discrete geodesic $\gamma(0, 1) = \{0 \rightarrow^n 1\} = \mathbb{Z} = \ldots, 1010, 010, 10, 0, 1, 01, 101, 0101, \ldots$

Call this *geodesimal notation*, tally notation with sign and parity bits. Geodesimal operations: $x \xrightarrow{3} y = yxy$, $x \xrightarrow{-3} y = yxyx$, etc.
8. The free space $1+1+1$. 3 connected components $0\alpha, 1\alpha, 2\alpha$

All points out to $\infty$ shown. Curvature $\kappa$ undefined ($-\infty$).

Triangles congruent by defn. but $\angle$, $\angle$, and $\angle$ incomparable.

$\exists$ disjoint inclined geodesics: $\gamma(101, 201) \cap \gamma(102, 202) = \emptyset$ (barely!)
9. The curvature hierarchy

All spaces (including $1 + 1 + 1$ itself) homogeneous.
Not shown: Sets ($xy = x$, §3), Dice ($xyxy = x$, §11).
10. Dice and subdirect irreducibles of Grv

The edge $E = E_3 = \{1, 0 = 2, 3\}$ is the unique geodesic with an odd number of points and two connected components.

- $E_3 = \mathbb{Z}_4/\{0 = 2\}$
- $E_6 = \mathbb{Z}_8/\{0 = 4, 2 = 6\}$
- $E_{12} = \mathbb{Z}_{16}/\{0 = 8, 2 = 10, 4 = 12, 6 = 14\}$, etc.

$Ab$ and $Grv$ have the same SI’s (subdirect irreducibles), namely $\mathbb{Z}_{p^n}$, $n \leq \infty$, as groves, except for $p = 2$ when $\mathbb{Z}_{4,2^n}$ is replaced by $E_{3,2^n}$ in Grv. ($\mathbb{Z}_{p^\infty}$ is the Prüfer $p$-group = the direct limit of the inclusion $\mathbb{Z}_{p^0} \subseteq \mathbb{Z}_{p^1} \subseteq \mathbb{Z}_{p^2} \subseteq \ldots$) Key fact: $\mathbb{Z}_4$ is a subdirect product of $E$’s.

$E \in \mathcal{V}$ iff $\mathbb{Z}_4 \in \mathcal{V}$ for all varieties $\mathcal{V} \subseteq Gsp$.

A die is a subspace of $E^n$, $n \leq \infty$. Equivalently, a model of $xx = xyy = x$, $xyxy = x$.

$Dice = HSP(\mathbb{Z}_4) = SP(E) \subset Grv$. 

11. The geodesic neighborhood

**Operations:** $xy$

$$\begin{align*}
\text{Gsp} & \quad \xrightarrow{[yxy]} \quad \text{Heap} & \quad \xrightarrow{yxz} & \quad \text{Grp} \\
\xrightarrow{x(\pi^2_1)} & \quad \xrightarrow{xyxz} & \quad \xrightarrow{yx} & \quad \xrightarrow{x^2} \\
\text{G3: } xywz = xzw & \quad [yxy] = [zyx] & \quad y - x + z & \quad x^n = x \\
\text{Grv} & \quad \xrightarrow{xyxz} & \quad \text{Schar} & \quad \xrightarrow{x - y} & \quad \text{Ring} \\
\xrightarrow{xy = x} & \quad [yxy] = x & \quad y + x + z & \quad x + y & \quad \text{Bool} \\
\xrightarrow{x(\pi^3_2)} & \quad \text{Set} & \quad \Cube & \quad \text{Cube}^* & \quad \text{Cube}^* \\
\end{align*}$$

Every path in this commutative diagram denotes a forgetful functor, hence one with a left adjoint. Vertical arrows *forget* the indicated *equation*, horizontal arrows *interpret* the blue *operation* above as the arrow’s label. E.g. the left adjoint of the functor $U_{\text{AbGrp}} : \text{Ab} \rightarrow \text{Grp}$ is abelianization, the arrow to Schar from Ab interprets Schar’s $[yxz]$ as $y - x + z$ in Ab, the left adjoint of the functor $U_{\text{SetGsp}} : \text{Set} \rightarrow \text{Gsp}$ gives the set $F_{\text{GspSet}}(S)$ of connected components of $S$, and so on.
12. Groves: \( Grv = Gsp + G3 \). Euclid’s 5th postulate

Euclid’s fifth or parallel postulate: \( EX \) and \( HY \), when inclined inwards, meet when \emph{produced}. Euclid: “inclined” = \( \alpha + \beta < 180^\circ \).

Our inclination condition: a \emph{witness triangle} \( \Delta AEH \) with parallelogram \( BCGF \) (centroid \( D \)) s.t \( B, C \) at midpoints of \( AE, AH \).

Our 5th postulate: \( EF \) and \( HG \), when obtained by extending the four sides of the skew quadrilateral \( ABDC \), meet when \emph{extended}.

\[
A \rightarrow B \rightarrow (C \rightarrow D) = A \rightarrow C \rightarrow (B \rightarrow D) \\
E \rightarrow F = H \rightarrow G
\]  

\( (G3) \)

G3 \( xy(zw) = xz(yw) \mid xywz = xzwy \mid xywzywz = x \mid x102102 = x \)