Paracompact box products

(again)

Judith Roitman
University of Kansas

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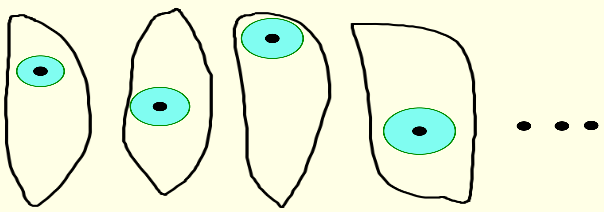
What's a box product?



Definition Box product: underlying set is a product of spaces $\Pi_{i \in I} X_i$; basic open set is a product of open sets $\Pi_{i \in I} u_i$.

Written $\Box_{i \in I} X_i$

The question

Question Which box products are paracompact? normal?

Definition X is paracompact iff every open cover has a locally finite open covering refinement.

For our purposes, suffices to consider

Definition X is ultraparacompact iff every open cover has a pairwise disjoint covering refinement.

Definition X is normal iff any two disjoint closed sets can be separated by disjoint open sets.

Paracompact vs. normal

All spaces Hausdorff.

Metrizable $\Rightarrow$ regular + paracompact $\Rightarrow$ normal.

Compact $\Rightarrow$ regular + paracompact $\Rightarrow$ normal.

All spaces regular.

General pattern

Negative results are in ZFC, prove non-normality.

Positive results are consistency results, prove paracompactness.

Where positive consistency results are known, we do not know independence.

Negative results

Theorem (Lawrence 1996) $\Box(\omega+1)^{\omega_1}$ not normal

I.e.,can't have uncountably many factors.

Theorem (a) (Kunen 1973) $\Box(2^{\mathfrak{c}^+})^{\omega}$ is not normal.

(b) (van Douwen 1977) $\Box(2^{\omega_2})^{\omega}$ is not normal.

I.e., need small character or small weight or some small base property...

Theorem (van Douwen 1975) $\mathbb{P} \times \Box(\omega+1)^{\omega}$ is not normal.

I.e., need compact or countably compact or some small covering property...

Historically first question (Tietze, 1940's) Is $\Box\mathbb{R}^\omega$ normal?

Historically second question (A. Stone, 1950's) Is the box product of countably many separable metrizable spaces normal?

First major result (M.E. Rudin, 1972) Assume CH. The box product of countably many compact metrizable spaces is paracompact.

More specific question Is the box product of countably many

(a) compact metric spaces

(b) compact first countable (every point has a countable neighborhood base) spaces

paracompact?

For (a), yes under many hypotheses. (Rudin 1972, Kunen 1978, van Douwen 1980)

For (b), yes under many hypotheses (van Douwen 1980 , JR 1979)

Two outliers

Theorem (Kunen 1978) The box product of countably many compact scattered spaces is consistently paracompact.

Theorem (Williams 1984) The box product of countably many compact spaces of weight $\leq \omega_1$ is paracompact.

Compact can be relaxed in various ways.

Theorem (Lawrence 1988) The box product of countably many countable metrizable spaces is consistently paracompact.

Theorem (Wingers 1994) The box product of countably many σ-compact 0-dimensional first countable spaces of cardinality $\leq \mathfrak{c}$ is consistently paracompact.

Towards a unified approach

Definition X is 0-dimensional iff it has a base of clopen sets.

Definition X is κ-open (a.k.a. a P_κ space) iff the intersection of fewer than κ open sets is open.

Definition X is κ-Lindelöf iff every open cover has a subcover of size $< \kappa$.

Theorem A 0-dimensional κ-open and κ^+-Lindelöf space is ultraparacompact.

Proof. Cover (WLOG) by clopen sets. There's a subcover $\{u_\alpha : \alpha < \lambda\}$ by no more than κ sets. Let $w_\alpha = u_\alpha \setminus \bigcup_{\beta<\alpha} u_\beta$. The w_α's give a disjoint open covering refinement.

Towards using this theorem

Definition $\nabla_{n<\omega} X_n = \square_{n<\omega} X_n / =^*$, where $x =^* y$ iff $\{n : x(n) \neq y(n)\}$ is finite.

$x_\nabla = [x]/ =^*$.

Fact $\nabla_{n<\omega} X_n$ is 0-dimensional.

Fact (Kunen 1978) If each X_n is locally compact, $\square_{n<\omega} X_n$ is paracompact iff $\nabla_{n<\omega} X_n$ is paracompact.

When is $\nabla_{n<\omega} X_n$ κ-open and κ^+-Lindelöf for some κ?

Fact

(a) If each X_n is first countable, $\nabla_{n<\omega} X_n$ is $\mathfrak{b}$-open. ($\mathfrak{b}$ is the smallest size of an unbounded family in $\omega^\omega / =^*$)

(b) If each X_n is second countable (e.g., compact metrizable), $\nabla_{n<\omega} X_n$ is $\mathfrak{d}^+$-Lindelöf. ($\mathfrak{d}$ is the smallest size of a dominating family in $\omega^\omega / =^*$)

Corollary If $\mathfrak{b} = \mathfrak{d}$ then the box product of countably many compact metrizable spaces is paracompact.

Recall A 0-dimensional κ-open and κ^+-Lindelöf space is ultraparacompact.

Definition X is basic κ-open iff it has a clopen base $\mathcal{B}$ so that the union of fewer than κ sets from $\mathcal{B}$ is closed.

Fact A basic κ-open κ^+-Lindelöf space is ultraparacompact.

Fact If each X_n is compact first countable, $\nabla_{n<\omega} X_n$ is basic $\mathfrak{d}$-open and $\mathfrak{c}^+$-Lindelöf.

Corollary if $\mathfrak{d} = \mathfrak{c}$ then the box product of countably many compact first countable spaces is paracompact.

Assume each X_n is first countable. Why is $\nabla_{n<\omega} X_n$ basic $\mathfrak{d}$-open?

Definition A box is a set of the form $B = \Pi_{n<\omega} B_n$.

Note that if each B_n is open, so is B; if each B_n is closed, so is B.

Let $\mathcal{D}$ be the set of all clopen countable intersections of boxes. $\mathcal{D}$ is a base witnessing basic $\mathfrak{d}$-open

Why? Because if you have a family of fewer than $\mathfrak{d}$ partial functions from ω to ω with infinite domain, there is one function in ω^ω which is not bounded (mod finite) by any of the partial functions on their domains.

What about not compact?

Without at least local compactness, can't use the ∇-product.

Instead of $\mathcal{D}$ a family of boxes, $\mathcal{D}$ is a *simple* family of boxes (if a tail of a point x is covered in a certain way, so is x_∇)

Fact A simple collection of fewer than $\mathfrak{d}$ closed boxes has closed union.

This fact allows us to adapt the proofs of the previous theorems to prove

(Wingers) The box product of countably many σ-compact 0-dimensional first countable spaces of size $\leq \mathfrak{c}$ is paracompact (if $\mathfrak{d} = \mathfrak{c}$)

To prove (Lawrence) $\Box\mathbb{Q}^\omega$ is paracompact (under $\mathfrak{b} = \mathfrak{d}$)we need more (simple tapered families, a tree structure on (some) points...).

Stacking up

So far the technique has been to take one set, then another, then another... and construct a pairwise disjoint refining cover by stage $\mathfrak{d}$.

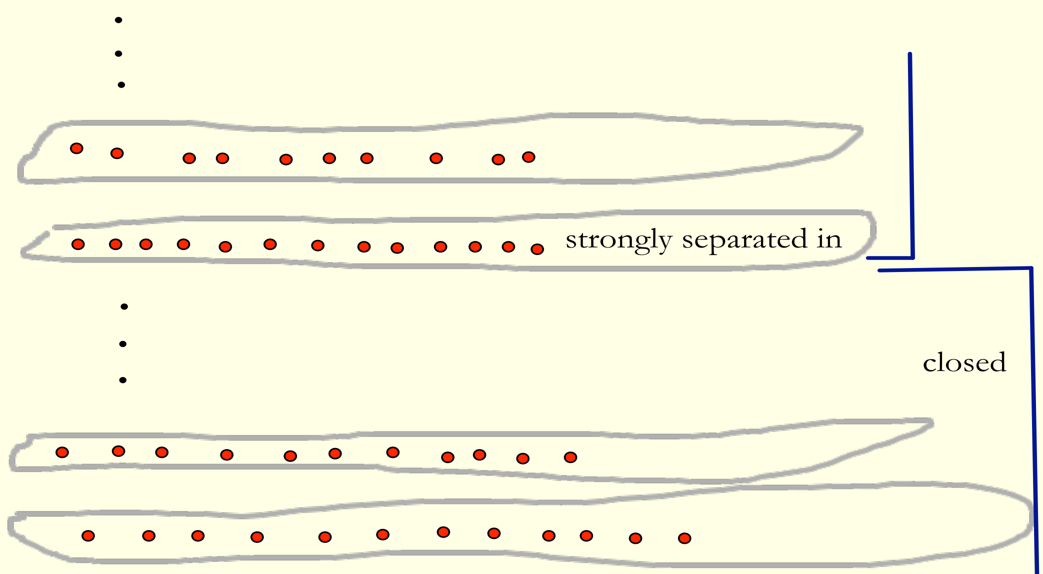
What if we take more than one set at a time?

<u>**Definition**</u> $\mathcal{E} \subset \wp(X)$ is a discrete collection iff no point in X is in the closure of more than one set in $\mathcal{E}$.

<u>**Definition**</u> Y is a strongly seperated subspace of X iff there is a discrete open collection $\mathcal{U} = \{u_y : y \in Y\}$ with $y \in u_y$ and if $y \neq y'$ then $u_y \neq u_{y'}$.

The idea is to layer strongly separated spaces so that witnessing separating families (a) refine the original cover, and (b) cover the whole space.

Stratification theorem If X is κ-open, 0-dimensional, $X = \bigcup_{\alpha<\kappa} X_\alpha$ where each $\bigcup_{\beta<\delta} X_\beta$ is closed, $\delta \leq \kappa$, and each X_α is strongly separated in $\bigcup_{\beta\geq\alpha} X_\beta$, then X is ultraparacompact.



[In fact the requirement of κ-open is a little stronger than needed.]

Definition 1. $H(\lambda)$ is the collection of all sets whose transitive closures have size $< \lambda$. 2. $H \prec_{\text{weak}} G$ iff $H \cap \wp(\omega)$ is nicely closed.

Model Hypothesis (MH) $H(\omega_1) = \bigcup_{\alpha<\kappa} H_\alpha$ where each $H_\alpha \prec_{\text{weak}} (H(\omega_1), \in)$ and each $H_\alpha \cap \omega^\omega$ is not dominating.

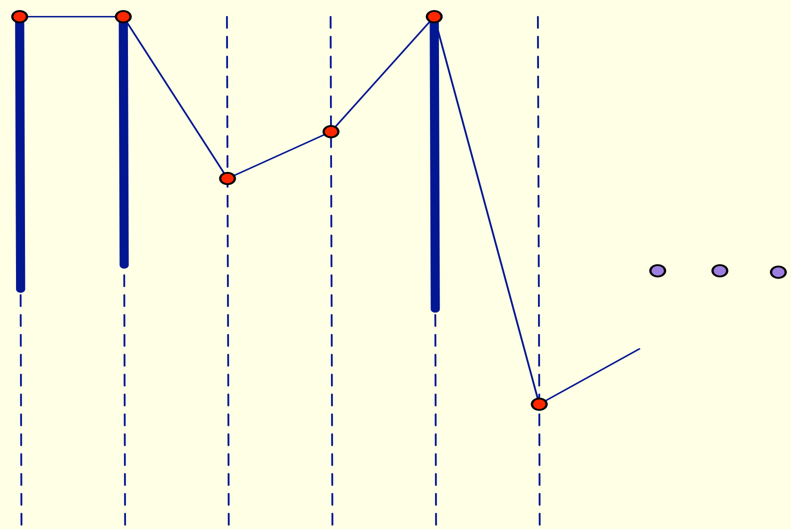
MH is implied by: $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{d} = \mathfrak{c}$ (hence MA); iterated ccc forcing of uncountable cardinality; Hechler iteration of Hechler forcing if cofinalities are uncountable; forcing with a measure algebra over a model of MH...

Fact MH can be used in place of $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{d} = \mathfrak{c}$ in preceding proofs with compact factors.

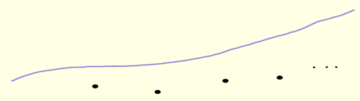
Sketch of proof Assume MH. $\nabla \subset H(\mathfrak{c})$, each $\nabla \cap \bigcup_{\beta<\alpha} H_\beta$ is closed, and each $\nabla \cap H_\alpha$ is strongly separated and closed in $\nabla \cap \bigcup_{\beta\geq\alpha} H_\beta$.

The most basic question

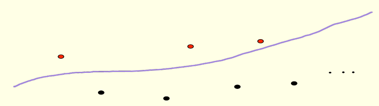
Question Is $\square(\omega+1)^\omega$ really paracompact?



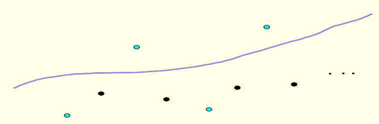
A point and a basic neighborhood:



A point and a basic neighborhood and a point in that neighborhood



A point and a basic neigbhborhood and a point not in that neighborhood



Easier question What subspaces of $\square(\omega+1)^\omega$ are really paracompact?

Question asked around 2005, answers quickly followed.

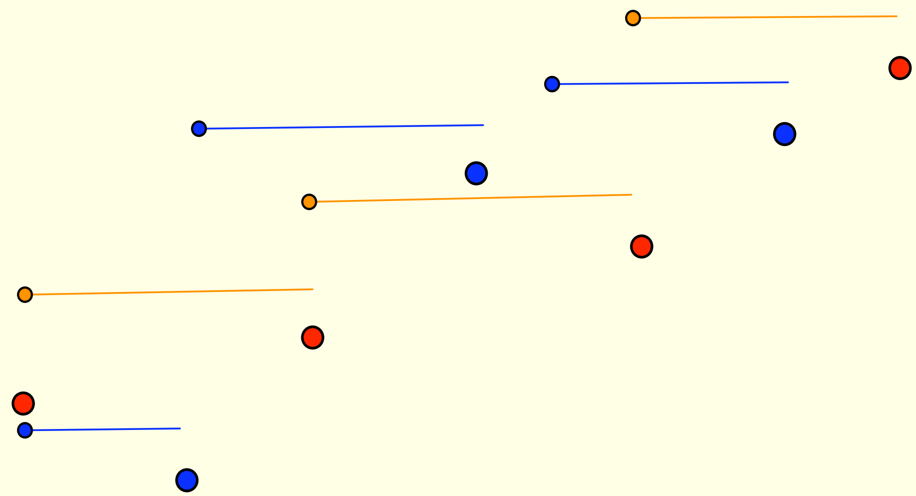
Notation: $\nabla = \nabla(\omega+1)^\omega, \square = \square(\omega+1)^\omega$.

Notation: By a partial function, we mean a function into ω whose domain is an infinite/co-infinite subset of ω.

Identify a point $x \in \square$ with the partial function $f_x = x^{\leftarrow}[\omega]$. Since $|\{x_\nabla : \text{dom } f_x \text{ is finite}\}| = 1$, and $\{x_\nabla : \omega\setminus \text{ dom } f_x \text{ is co-finite}\}$ is discrete, all we care about are the x_∇ for which f_x is a partial function.

Theorem Let $Y = \{f_\triangledown : f$ is an increasing partial function$\}$. Y is closed discrete (i.e., $\{\{f_\triangledown\} : f_\triangledown \in Y\}$ is discrete.)

Proof.



Definition $cn(X)$ is the least κ so there is a clopen base $\mathcal{B}$ with the union of $< \kappa$ sets in $\mathcal{B}$ is closed.

Definition Let X be a space, $\preceq$ a pre-order on X, $Y \subseteq X$. MOH$(Y, \preceq)$ is the following statement: $(Y, \preceq)$ is a tree, and $\forall y \in Y \ u_y = \{z \in Y : y \preceq z\}$ is open in X.

Theorem If X is 0-dimensional, MOH$(Y, \preceq)$ and ht$(Y) \leq cn(X)$, then Y is ultraparacompact.

Proof By MOH, Y can be stratified.

GMOH$(X, \preceq)$ is the statement that any $\approx_{\preceq}$ transversal of X satisfies MOH.

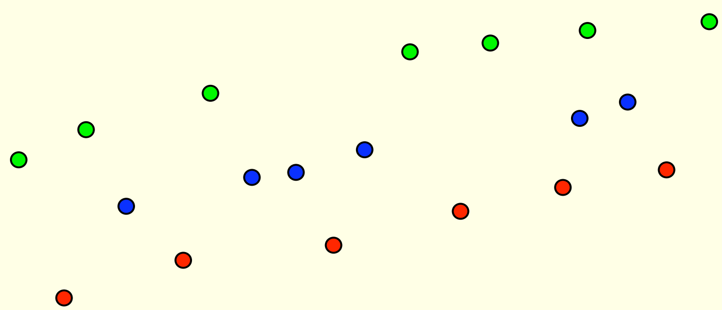
Theorem If $\preceq$ is a pre-order on $\square$ coarser than $\leq^*$ satisfying GMOH then any $\approx_\preceq$ transversal is ultraparacompact.

Such pre-orders are not difficult to find.

First example

Definition Given a partial function f with domain a,
$\perp(f) = \{n \in a : \forall m > n \text{ if } m \in a \text{ then } f(m) \geq f(n)\}$. $f^{\perp} = f|_{\perp(f)}$.

Definition $f_0 = f^{\perp}$; for each n, $f_n = (f \setminus \bigcup_{m \leq n} f_n)$.



Definition $ht(f) = n$ iff n is least so f_{n+1} is finite. $ht(f) = \omega$ iff $\forall n \; f_n$ is infinite.

Example $f \prec_\perp g$ iff $\forall n \leq ht(f) \; f_n =^* g_n$.

Fact $\prec_\perp$ satisfies GMOH. Hence any $\approx_{\preceq_\perp}$ transversal is paracompact.

This result can be extended to include some subsets of $\{f : ht(f) = \omega\}$.

Second example

Fix $\vec{h} = \{h_\alpha : \alpha < \mathfrak{b}\}$ unbounded, well-ordered by $\leq^*$, each h_α is increasing.

Definition Given a partial function f and $\alpha < \kappa$, $a_{f,\alpha} = \{n : f(n) < h_\alpha(n)\}$.

Definition $f_0 = f|_{a_{f,0}}$ if $a_{f,0}$ is infinite. Otherwise $f_0 = \emptyset$. For $\alpha > 0$, $f_\alpha = f|_{a_{f \setminus \bigcup_{\beta<\alpha} f_\beta, \alpha}}$ if $a_{f \setminus \bigcup_{\beta<\alpha} f_\beta}$ is infinite. Otherwise $f_\alpha = \emptyset$.

Definition $E(f) = \{\alpha : f_\alpha \neq \emptyset\}$.

Example $f \preceq_{\vec{h}} g$ iff $E(g)$ is an end-extension of $E(f)$ and $\forall \alpha \in E(f)\ f_\alpha =^* g_\alpha$.

Fact $\prec_{\vec{h}}$ satisfies GMOH. Hence any $\approx_{\preceq_{\vec{h}}}$ transversal is paracompact.

Extensions

1. The approach used in $\preceq_{\vec{h}}$ can be used to find paracompact subspaces of box products of countably many countable metrizable factors if $\mathfrak{b} = \mathfrak{d}$.

Sketch of proof $\preceq_{\vec{h}}$ gives a tree structure with the necessary properties.

2. The pre-order $\preceq_{\vec{h}}$ can be refined to get more pre-orders satisfying GMOH

One last combinatorial principle

Definition Δ is the following statement: for all partial functions f there is a total function x_f so if $f \setminus g$ and $g \setminus f$ are infinite and f, g are compatible, then either $x_f|_{\text{dom }(g\setminus f)} \not\leq^* (g \setminus f)|_{\text{dom }(g\setminus f)}$ or $x_g|_{\text{dom }(f\setminus g)} \not\leq^* (f \setminus g)|_{\text{dom }(f\setminus g)}$.

Δ holds if $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{d} = \mathfrak{c}$ or MH. Also, it cannot be destroyed by forcing with a measure algebra.

Theorem If Δ holds, then ∇ is ultraparacompact.

Theorem If Δ holds, then ∇ is ultraparacompact.

Sketch of proof 1. Let $\{f_\alpha : \alpha < \omega\}$ be a family of partial functions so every partial function is almost contained in some f_α.

2. $\nabla_\alpha = \{f : \alpha \text{ is least with } f \subseteq^* f_\alpha\}$.

3. Each ∇_α is strongly separated.

4. Given an open cover $\mathcal{U}$ of ∇, first refine to separate each ∇_α.

5. Then refine using the functions x_{f_α} witnessing Δ.

6.. Look carefully at the combinatorics. Cover in stages whatever hasn't been covered before. Done.

What we should know but don't

1. Is $\square(\omega+1)^\omega$ really paracompact?

2. What about the other positive consistency results? Independent or real?

3. Don't forget Tietze's $\square\mathbb{R}^\omega$. Even consistency would be good here.

4. Is there a box product with infinitely many nice factors which is really normal but consistently not paracompact?