Cardinal Invariants of Group Actions

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FIELDS

The family of cardinals cov

- In general, if $\mathcal{J}$ is any ideal then $\mathbf{cov}(\mathcal{J})$ will denote the least cardinal of a subset $\mathcal{A} \subseteq \mathcal{J}$ such that $\cup\mathcal{A} = \cup\mathcal{J}$.
- In particular, if X is a Polish space and the ideal of meagre sets in X is denoted by $\mathcal{M}_X$ and $\mathbf{cov}(\mathcal{M}_X)$ will denote the least cardinal of a subset $\mathcal{A} \subseteq \mathcal{M}_X$ such that $X = \cup\mathcal{M}_X$.
- If G is a locally compact group and the ideal of Haar null sets in G is denoted by $\mathcal{N}_G$ and $\mathbf{cov}(\mathcal{N}_G)$ will denote the least cardinal of a subset $\mathcal{A} \subseteq \mathcal{N}_G$ such that $G = \cup\mathcal{N}_G$.

It turns out that the general setting implied by these definitions is misleading since $\mathbf{cov}(\mathcal{M}_X) = \mathbf{cov}(\mathcal{M}_{\mathbb{R}})$ for all Polish spaces without isolated points. Moreover, $\mathbf{cov}(\mathcal{N}_G) = \mathbf{cov}(\mathcal{N}_{\mathbb{R}})$ for all locally compact, second countable, non-discrete groups.

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THE CARDINALS non, cof AND add

Similar observations show that other cardinals are also not sensitive to the underlying structure defining them. Recall that if $\mathcal{J}$ is any ideal then

- **add**$(\mathcal{J})$ denotes the least cardinal of a subset $\mathcal{A} \subseteq \mathcal{J}$ such that $\cup\mathcal{A} \notin \mathcal{J}$
- **non**$(\mathcal{J})$ denotes the least cardinal of a subset $A \subseteq \cup\mathcal{J}$ such that $A \notin \mathcal{J}$
- **cof**$(\mathcal{J})$ denotes the least cardinal of a subset $\mathcal{A} \subseteq \mathcal{J}$ such that for all $B \in \mathcal{J}$ there is $A \in \mathcal{A}$ such that $B \subseteq A$.

The cardinals **add**$(\mathcal{M})$, **non**$(\mathcal{M})$ and **cof**$(\mathcal{M})$ do not depend on the space in which the ideal of meagre sets $\mathcal{M}$ is defined (as long as the space is Polish without isolated points). Similar remarks hold for the ideal of sets of Haar measure zero.

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In order to have the familiar cardinals associated with the continuum provide more information about the underlying spaces used in their definition one can add group structure to the definitions. The following definitions can be found in *Bartoszynski–Judah*. Let $\mathcal{J}$ be an ideal on a group $(G, +)$.

- **add**$^*(\mathcal{J})$ denotes the least cardinal of a subset $A \subseteq G$ such that there is $B \in \mathcal{J}$ such that $A + B = \{a + b \mid a \in A \text{ and } b \in B\} \notin \mathcal{J}$
- **cov**$^*(\mathcal{J})$ denotes the least cardinal of a subset $A \subseteq G$ such that there is $B \in \mathcal{J}$ such that $A + B = G$
- **cof**$^*(\mathcal{J})$ denotes the least cardinal of a subset $\mathcal{A} \subseteq \mathcal{J}$ such that for all $B \in \mathcal{J}$ there is $A \in \mathcal{A}$ and $g \in G$ such that $B \subseteq g + A$.

The cardinal **non** does not fit well into the transitive scheme. Notice also that in the non-commutative case one would have to consider left and right cardinals.

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The transitive cardinals $\mathbf{add}^*$ and $\mathbf{cof}^*$

In the following theorems of Pawlikowski, G can be $\mathbb{R}$, the circle group or a countable product of finite abelian groups.

Theorem (Pawlikowski)

$\mathbf{add}(\mathcal{M}) = \min(\mathbf{add}^*(\mathcal{M}_G), \mathfrak{b})$.

Theorem (Pawlikowski)

$\mathbf{add}(\mathcal{N}) = \min(\mathbf{add}^*(\mathcal{N}_G), \mathfrak{b})$.

Theorem (Pawlikowski)

$\mathbf{cof}^*(\mathcal{M}_G) = \mathfrak{d}$.

Theorem (Pawlikowski)

$\mathbf{cof}^*(\mathcal{N}_G) = \mathbf{cof}(\mathcal{N})$.

The transitive cardinal cov*

Recall that the cardinal **cov**$(\mathcal{M})$ can be characterized in terms of eventually different reals:

Theorem (Bartoszynski, Miller)

cov$(\mathcal{M})$ *is equal to the least cardinal of an eventually different family, in other words, a family* $\mathcal{E} \subseteq \mathbb{N}^{\mathbb{N}}$ *such that for each* $f : \mathbb{N} \to \mathbb{N}$ *there is* $e \in \mathcal{E}$ *such that* $e(n) \neq f(n)$ *for all* $n \in \mathbb{N}$.

An analogue of this theorem exists for the cardinal **cov***.

Theorem

In the realm of $\mathbb{R}$*, the circle group, or a countable product of finite groups the cardinal* **cov**$^*(\mathcal{M})$ *is equal to the least cardinal of a bounded eventually different family.*

However, it will be seen that the cardinal **cov*** does depend on G, so the notation of *Bartoszynski–Judah* will be abandoned for more precise notation.

Definition

Let X be a Polish space, $\mathbb{G} = (G, \cdot)$ a group, $\mathcal{J} \subseteq \mathcal{P}(X)$ (usually an ideal) and $\mathcal{F} \subseteq \mathcal{J}$. Let $\alpha : G \times X \to X$ be a group action of $\mathbb{G}$ on X. Then

- $\mathbf{add}_\alpha(\mathcal{J}, \mathcal{F})$ denotes the least cardinal of a subset $A \subseteq G$ such that there is $B \in \mathcal{J}$ such that $\alpha(A \times B) \notin \mathcal{F}$
- $\mathbf{cov}_\alpha(\mathcal{J})$ denotes the least cardinal of a subset $A \subseteq G$ such that there is $B \in \mathcal{J}$ such that $\alpha(A \times B) = G$
- $\mathbf{cof}_\alpha(\mathcal{J}, \mathcal{F})$ denotes the least cardinal of a subset $\mathcal{A} \subseteq \mathcal{J}$ such that for all $B \in \mathcal{J}$ there is $A \in \mathcal{A}$ and $g \in G$ such that $B \subseteq \alpha(g, A)$.

In the important case that α is the action of the group on itself then the notation $\mathbf{add}_\mathbb{G}$, $\mathbf{cov}_\mathbb{G}$ and $\mathbf{cov}_\mathbb{G}$ will be used instead. If $\mathcal{J} = \mathcal{F}$ then $\mathbf{add}_\alpha(\mathcal{J})$ will be used instead of $\mathbf{add}_\alpha(\mathcal{J}, \mathcal{F})$.

Examples of actions and cardinals

- If $\mathcal{J}$ is either $\mathcal{M}$ or $\mathcal{N}$ (or any other ideal for that matter) and $\mathcal{F}$ is the set of all singletons in $\mathbb{R}$ then $\mathbf{add}_{\mathbb{R}}(\mathcal{J}, \mathcal{F}) = \mathbf{non}(\mathcal{J})$
- If $\mathcal{F}$ is the set of circles in $\mathbb{R}^2$ then $\mathbf{add}_{\mathbb{R}^2}(\mathcal{N}, \mathcal{F})$ will be considered later.
- If α is the action of the isometry group of $\mathbb{R}^n$ on $\mathbb{R}^n$ then $\mathbf{add}_{\alpha}(\mathcal{N}, \{\mathbb{R}^k\})$ will be considered as well.
- The cardinal $\mathbf{cof}_{\alpha}(\mathcal{J}, \mathcal{F})$ can also be interesting when $\mathcal{F}$ is a singleton.
- The cardinal $\mathbf{cov}_{\alpha}(\mathcal{J})$ can be of interest when $\mathcal{J}$ is the ideal generated by a single set — for example, $\mathbf{cov}_{\mathbb{R}}(\{C\})$ will be examined later for particular sets $C \subseteq \mathbb{R}$.

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Cardinals specific to group actions

Some cardinals only make sense in the context of a group action.

Definition

For group $\mathbb{G}$ and $A \subseteq G$ the packing index is
$pack(A) = \sup\{|X| \mid \{xA \mid x \in X\}$ is disjoint$\}$. but one might also consider the minimal packing index is
$minpack(A) = \inf\{|X| \mid \{xA \mid x \in X\}$ is maximal disjoint$\}$.

It is possible to use construct sets A with prescribed packing indices by using Choice — so these questions are mostly of interest for Borel sets A.

Question (Banakh & Lyaskovska)

Is there a Polish group $\mathbb{G}$ and Borel $A \subseteq G$ such that $\aleph_0 < pack(A) < 2^{\aleph_0}$?

$\mathbf{cov}_{\mathbb{R}}(\{C\})$

Gruenhage observed that if C is the standard Cantor set then $\mathbf{cov}_{\mathbb{R}}(\{C\}) = 2^{\aleph_0}$ and he asked:

Question (Gruenhage)

Is there any compact Lebesgue null set $A \subseteq \mathbb{R}$ such that it is consistent that $\mathbf{cov}_{\mathbb{R}}(\{A\}) < 2^{\aleph_0}$?

A similar question was asked by Mauldin:

Question (Mauldin)

Is there any compact set $A \subseteq \mathbb{R}$ of Hausdorff dimension less one than such that it is consistent that $\mathbf{cov}_{\mathbb{R}}(\{A\}) < 2^{\aleph_0}$?

These questions have to be asked in precise contexts for in broad generality the following result will provide answers:

Theorem (Solecki)

For a large class of abelian groups $\mathbb{G}$ with left invariant measure μ there is a left invariant extension ν such that $\mathbf{cov}_{\mathbb{G}}(\mathcal{N}_\nu) = \aleph_1$.

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The following is a partial response to the question of Mauldin:

THEOREM (DARJI & KELETI)

If $\mathbb{GL}$ is the linear group on $\mathbb{R}$ and $A \subseteq \mathbb{R}$ has packing dimension less than one then $\mathbf{cov}_{\mathbb{GL}}(\{A\}) = 2^{\aleph_0}$.

Darji and Keleti also noted that the equality $\mathbf{cov}_{\alpha}(\{A\}) = 2^{\aleph_0}$ for a $\mathbb{G}$ action α on $\mathbb{R}$ would follow from the existence of a perfect set $P \subseteq \mathbb{R}$ such that $\alpha(\{g\} \times A) \cap P$ is countable for all $g \in \mathbb{G}$ and asked whether such a P always can be found.

Definition (Erdös & Kakutani)

$$C_{EK} = \left\{ \sum_{n=2}^{\infty} \frac{d_n}{n!} \mid 0 \leq d_n \leq n-2 \right\}$$

Note that all but countably many $x \in [0,1]$ have a unique representation as

$$x = \sum_{n=2}^{\infty} \frac{x_n}{n!}$$

where $0 \leq x_n \leq n-1$ for each n.

Theorem (Elekes & S)

For all perfect $P \subseteq \mathbb{R}$ there is x such that $(x + C_{EK}) \cap P$ is uncountable.

This answer's the Darji–Keleti question. Moreover ...

…it is also true that:

Theorem (Elekes & S)

$\mathbf{cov}_{\mathbb{R}}(\{C_{EK}\}) \leq \mathbf{cof}(\mathcal{N})$. *(Recall that $\mathcal{N}$ is the ideal of Lebesgue null sets.)*

and hence it is consistent that $\mathbf{cov}_{\mathbb{R}}(\{C_{EK}\}) < 2^{\aleph_0}$, answering Gruenhage's question. This result can be generalized considerably. For a locally compact group $\mathbb{G}$ let $\mathcal{H}_{\mathbb{G}}$ be ideal of compact subsets of $\mathbb{G}$ of Haar measure zero.

Theorem (Elekes & Tóth)

If $\mathbb{G}$ is uncountable, compact and abelian or if $\mathbb{G}$ is non-discrete, separable, locally compact and abelian then $\mathbf{cov}_{\mathbb{G}}(\mathcal{H}_{\mathbb{G}}) \leq \mathbf{cof}(\mathcal{N})$.

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A complete characterization of the inequality $\mathbf{cov}_{\mathbb{G}}(\mathcal{H}_{\mathbb{G}}) < 2^{\aleph_0}$ is available. Recall that a group is profinite if it is the inverse limit of finite groups. Being the inverse limit of compact groups, these have a natural compact topology.

Theorem (Elekes & Tóth)

If $\mathbf{cov}_{\mathbb{G}}(\mathcal{H}_{\mathbb{G}}) < 2^{\aleph_0}$ for every profinite group $\mathbb{G}$ then for any locally compact group $\mathbb{G}$ the following are equivalent:

- $\mathbf{cov}_{\mathbb{G}}(\mathcal{H}_{\mathbb{G}}) < 2^{\aleph_0}$
- *$\mathbb{G}$ is non-discrete and has no open subgroup of index $2^{\aleph_0}$.*

Theorem (Abért)

It is consistent with set theory that $\mathbf{cov}_{\mathbb{G}}(\mathcal{H}_{\mathbb{G}}) < 2^{\aleph_0}$ for every profinite group $\mathbb{G}$.

What Abért's proof shows

Recall that if $\{X_j\}_{j=0}^{\infty}$ are sets and $f : \mathbb{N} \to \mathbb{N}$ then an f-slalom is a set of the form $\prod_{j=0}^{\infty} S_j$ where $|S_j| = f(j)$. The cardinal $c(g, f)$ is the least number of f-slaloms needed to cover $\prod_{j=0}^{\infty} g(j)$.

Goldstern and Shelah proved it consistent that many $c(g, f)$ are all distinct and less than $2^{\aleph_0}$.

Given a profinite group $\mathbb{G}$ which is the inverse limit of $\{\mathbb{G}_j\}_{j=0}^{\infty}$ where $|\mathbb{G}_j| = g(j)$ Abért's proof actually yields for each $f : \mathbb{N} \to \mathbb{N}$ a compact set of Haar measure zero C_f such that $\mathbf{cov}_{\mathbb{G}}(\{C_f\}) \leq c(g, f)$. But this does not answer the following question:

Question

Is it consistent that there are groups $\mathbb{G}_0$ and $\mathbb{G}_1$ such that $\mathbf{cov}_{\mathbb{G}}(\mathcal{H}_{\mathbb{G}_0}) < \mathbf{cov}_{\mathbb{G}}(\mathcal{H}_{\mathbb{G}_1}) < 2^{\aleph_0}$?

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The body of work on covering by slaloms can be viewed in the context of covering numbers of group actions. Let $\mathbb{S}_k$ be the symmetric group on k and let α the natural action of $\prod_{j=0}^{\infty} \mathbb{S}_{g(k)}$ on $\prod_{j=0}^{\infty} g(k)$. Then $c(g, f) = \mathbf{cov}_\alpha(\{\prod_{j=0}^{\infty} f(k)\})$.
The appearance of $\mathbf{cof}(\mathcal{N})$ in the preceding theorems on $\mathbf{cov}_{\mathbb{G}}(\{C\})$ is partially explained by the following:

Theorem (Gruenhage & Levy)

If $\lim_{n\to\infty} f(n) = \infty$ and α is the natural action of $\mathbb{S}_{Fin} = \bigcup_{k=2}^{\infty} \mathbb{S}_k$ on $\mathbb{N}^{\mathbb{N}}$ then $\mathbf{cov}_\alpha(\{\prod_{j=0}^{\infty} f(j)\}) = \mathbf{cof}(\mathcal{N})$.

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Groups that are not locally compact

Groups that are not locally compact have no Haar measure so $\mathbf{cov}_{\mathbb{G}}(\mathcal{H}_{\mathbb{G}})$ is not meaningful. However, recall that even if $\mathbb{G}$ is not locally compact the following definition is useful:

Definition (Christensen)

A set $X \subseteq \mathbb{G}$ is Haar null if and only if there is a universally measurable set $A \subseteq \mathbb{G}$ such that $X \subseteq A$ and a Borel probability measure μ on $\mathbb{G}$ such that $\mu(gAh) = 0$ for all g and h in $\mathbb{G}$.

Let $\mathcal{HN}_{\mathbb{G}}$ be the ideal of Haar null sets on $\mathbb{G}$.

Theorem (Solecki)

For any non-locally compact Polish group with an invariant metric

- $\mathbf{add}(\mathcal{HN}_{\mathbb{G}}) \leq \mathfrak{b}$
- $\mathbf{cof}(\mathcal{HN}_{\mathbb{G}}) \geq \mathfrak{d}$

Note that $\mathbf{add}(\mathcal{HN}_{\mathbb{G}}) \leq \mathbf{add}_{\mathbb{G}}(\mathcal{HN}_{\mathbb{G}})$ and $\mathbf{cof}(\mathcal{HN}_{\mathbb{G}}) \leq \mathbf{cof}_{\mathbb{G}}(\mathcal{HN}_{\mathbb{G}})$ so Solecki's theorem yields useful information only about $\mathbf{cof}_{\mathbb{G}}(\mathcal{HN}_{\mathbb{G}})$.

Question

Is $\mathbf{add}_{\mathbb{G}}(\mathcal{HN}_{\mathbb{G}}) \leq \mathfrak{b}$?

Question

What is $\mathbf{cov}_{\mathbb{G}}(\mathcal{HN}_{\mathbb{G}})$?

Question

Is it consistent that $\mathbf{cof}_{\mathbb{G}}(\mathcal{HN}_{\mathbb{G}}) < 2^{\aleph_0}$?

But one can also consider the meagre ideal on groups that are not locally compact.

Theorem (Miller & S)

- *If $\mathbb{G}$ is an arbitrary group, X is σ-compact, second countable without isolated points and $\alpha : G \times X \to X$ is a $\mathbb{G}$-action all of whose orbits are dense and such that $\alpha(g, \cdot)$ is continuous then* $\mathbf{cov}_\alpha(\mathcal{M}_X) = \mathbf{cov}_\alpha(\mathcal{CND}_X)$.
- *If $\mathbb{G}$ is Polish and non-discrete then* $\mathbf{cov}_{\mathbb{G}}(\mathcal{M}_{\mathbb{G}}) = \mathbf{cov}_{\mathbb{G}}(\mathcal{CND}_{\mathbb{G}})$.

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In many cases the cardinal $\mathbf{cov}_{\mathbb{G}}(\mathcal{M}_{\mathbb{G}})$ yields nothing new.

Theorem (Miller & S)

Let $\mathbb{G}$ be a Polish group such that there are sets $\{B_k, A_k^j\}_{j,k\in\mathbb{N}}$ such that:

1. *there is an infinite branching tree $T \subseteq {}^{\mathbb{N}\smile}\mathbb{N}$ such that $\bigcap_{k=0}^{\infty} A_k^{b(k)} \neq \emptyset$ for $b \in \overline{T}$*
2. *$\bigcup_{k=0}^{\infty} B_k$ is dense open*
3. *$(A_k^i \cdot B_k) \cap (A_k^j \cdot B_k) = \emptyset$ unless $i = j$*

then $\mathbf{cov}_{\mathbb{G}}(\mathcal{M}_{\mathbb{G}}) = \mathbf{cov}(\mathcal{M})$.

Corollary

If $\mathbb{G}$ is either $\mathbb{Z}^{\mathbb{N}}$, the symmetric group on $\mathbb{N}$, the homeomorphism group of $[0,1]^{\kappa}$ or its boundary where $\kappa \leq \omega$, or the additive group of a Banach space with an unconditional basis then $\mathbf{cov}_{\mathbb{G}}(\mathcal{M}_{\mathbb{G}}) = \mathbf{cov}(\mathcal{M})$.

Dobrowolski and Marciszewski extended this theorem:

Theorem (Dobrowolski & Marciszewski)

If $\mathbb{G}$ is a Polish group that is not locally compact, but has left-invariant, complete metric then $\mathbf{cov}_{\mathbb{G}}(\mathcal{M}_{\mathbb{G}}) = \mathbf{cov}(\mathcal{M})$.

All Banach spaces have such a metric, but observe that the homeomorphism group $[0, 1]$ does *not* have a left-invariant, complete metric. Neverheless, they also made good use of the following, which they attribute to Banakh:

Theorem (Banakh)

If $\mathbb{G}_0$ is a closed subgroup of the Polish group $\mathbb{G}_1$ then $\mathbf{cov}_{\mathbb{G}_1}(\mathcal{M}_{\mathbb{G}_1}) \leq \mathbf{cov}_{\mathbb{G}_0}(\mathcal{M}_{\mathbb{G}_0})$.

Note that this implies that if $\mathbf{cov}_{\mathbb{G}_0}(\mathcal{M}_{\mathbb{G}_0}) = \mathbf{cov}(\mathcal{M})$ then $\mathbf{cov}_{\mathbb{G}_1}(\mathcal{M}_{\mathbb{G}_1}) = \mathbf{cov}(\mathcal{M})$. In particular, if $\mathbb{Z}^{\mathbb{N}}$ is a (necessarily closed) subgroup of $\mathbb{G}$ then $\mathbf{cov}_{\mathbb{G}}(\mathcal{M}_{\mathbb{G}}) = \mathbf{cov}(\mathcal{M})$.

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As a corollary to this, and the fact that $\mathbf{cov}_{\mathbb{Z}^{\mathbb{N}}}(\mathcal{M}_{\mathbb{Z}^{\mathbb{N}}}) = \mathbf{cov}(\mathcal{M})$, they obtain:

Corollary (Dobrowolski & Marciszewski)

If $\mathbb{G}$ is either

- *the group of order preserving bijections of $\mathbb{Q}$ with the topology of pointwise convergence*
- *the homeomorphism group of an n-manifold*
- *the homeomorphism group of a Hilbert cube manifold*
- *the homeomorphism group of the Cantor set*
- *the symmetric group on $\mathbb{N}$ with the topology of pointwise convergence*

then $\mathbf{cov}_{\mathbb{G}}(\mathcal{M}_{\mathbb{G}}) = \mathbf{cov}(\mathcal{M})$.

Of course, this also holds for any group containing one of these as a closed subgroup.

Recall that

Theorem (Bartoszynski, Miller)

$\mathbf{cov}(\mathcal{M})$ is equal to the least cardinal of an eventually different family, in other words, a family $\mathcal{E} \subseteq \mathbb{N}^{\mathbb{N}}$ such that for each $f : \mathbb{N} \to \mathbb{N}$ there is $e \in \mathcal{E}$ such that $e(n) \neq f(n)$ for all $n \in \mathbb{N}$.

Theorem

For $\mathbb{G}$ equal to $\mathbb{R}$, the circle group, or a countable product of finite groups the cardinal $\mathbf{cov}_{\mathbb{G}}(\mathcal{M}_{\mathbb{G}})$ is equal to $\mathfrak{eq}$, the least cardinal of a bounded eventually different family.

This can be extended to other groups such as $\mathbb{R}^n$ and $\mathbb{R}^n/\mathbb{Z}^n$. Recall that it is consistent that $\mathfrak{eq} > \mathbf{cov}(\mathcal{M})$.

Theorem (Miller & S)

It is relatively consistent with set theory that the continuum be $\aleph_2$, for every infinite compact group $\mathbf{cov}_{\mathbb{G}}(\mathcal{M}_{\mathbb{G}}) = \aleph_2$, and $\mathbf{cov}(\mathcal{M}) = \aleph_1$ (in fact, $\mathfrak{d} = \omega_1$).

Question

Is $\mathbf{cov}_{\mathbb{G}}(\mathcal{M}_{\mathbb{G}}) \geq \mathfrak{eq}$ *for every infinite compact group* $\mathbb{G}$*?*

Question

Is it consistent that $\mathbf{cov}_{\mathbb{G}}(\mathcal{M}_{\mathbb{G}}) > \mathfrak{eq}$ *for some infinite compact group* $\mathbb{G}$*?*

Question

Is it true that $\mathbf{cov}_{\mathbb{G}}(\mathcal{M}_{\mathbb{G}}) \in \{\mathbf{cov}(\mathcal{M}), \mathfrak{eq}\}$ *for any non-discrete Polish group* $\mathbb{G}$*?*

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Recall that $\mathbf{add}_\alpha(\mathcal{J}, \mathcal{F})$ denotes the least cardinal of a subset $A \subseteq G$ such that there is $B \in \mathcal{J}$ such that $\alpha(A \times B) \notin \mathcal{F}$. Let α_n the natural action of the isometry group on $\mathbb{R}^n$.

THEOREM (S)

- *Let P be a 2-dimensional plane in $\mathbb{R}^3$. It is consistent that $\mathbf{add}_{\alpha_3}(\{P\}, \mathcal{N}) = \aleph_1 < \mathbf{non}(\mathcal{N}) = \aleph_2$.*
- *Let $\mathcal{C}$ be the family of circles in $\mathbb{R}^2$ centred at the origin. It is consistent that $\mathbf{add}_{\alpha_2}(\mathcal{C}, \mathcal{N}) = \aleph_1 < \mathbf{non}(\mathcal{N}) = \aleph_2$.*
- *Let E be an ellipsoid in $\mathbb{R}^n$ and $\mathcal{E} = \{rE \mid r \in [0, 1]\}$. It is consistent that $\mathbf{add}_{\alpha_n}(\mathcal{E}, \mathcal{N}) = \aleph_1 < \mathbf{non}(\mathcal{N}) = \aleph_2$.*

While the preceding theorem applies to many other families, the following remains open:

Question

Let H be a helix in $\mathbb{R}^3$. Is it consistent that $\mathbf{add}_{\alpha_3}(\{H\},\mathcal{N}) = \aleph_1 < \mathbf{non}(\mathcal{N}) = \aleph_2$?

Question

Is it consistent that $\mathbf{add}_{\alpha_3}(\{P\},\mathcal{N}) \neq \mathbf{add}_{\alpha_n}(\mathcal{E},\mathcal{N})$?

It should be observed that a geometric duality argument solves the question if $\mathcal{E}$ is replaced with $\mathcal{C}$ and P by a line in $\mathbb{R}^2$.

Question

It is consistent that $\mathbf{add}_{\alpha_3}(\{P\},\mathcal{M}) = \aleph_1 < \mathbf{non}(\mathcal{M}) = \aleph_2$?

The same question is of interest for $\mathcal{C}$ and other families.

The proofs of the preceding results rely on the boundedness of certain maximal functions. The idea it to gain information about measures of sets in $\mathbb{R}^3$ from information about the measures of the sets restricted to dimensional spaces, such as spehere. The following Question FC (10£) on Fremlin's list is in this spirit:

Question (Fremlin)

Let λ be Lebesgue measure on $\mathbb{R}$ and μ be the natural measure on the Cantor set. Is there a set A such that $\lambda^*(A) > 0$ yet $\mu(A + x) = 0$ for all $x \in \mathbb{R}$?

The answer is positive if $\mathbf{cov}(\mathcal{N}) = \mathbf{cof}(\mathcal{N})$ or if $\mathbf{non}(\mathcal{N}) = 2^{\aleph_0}$ or if the middle thirds Cantor set is replaced by the middle three fifths Cantor set.

The following is a step towards a consistent negative answer to the category version of Fremlin's question:

Theorem (Bartoszynski)

In the model obtaned by iterating ω_2 Laver reals every second category set in $\mathbb{N}^{\mathbb{N}}$ has second category intersection with some compact set.

However, the set in this result may depend on the second category set.

Theorem (Elekes & S)

It is consistent that for every second category set $X \subseteq \mathbb{R}$ there is $x \in \mathbb{R}$ such that $X \cap (C_{EK} + x)$ is also second category.

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