A SIMPLE SOLUTION TO BURSIDE'S PROBLEM

These are my (= KK's) notes on a solution to Burnside's Problem discovered by Jan-Christoph Schlage-Puchta.

Here $\langle R\rangle_F$ denotes the normal subgroup of F generated by R.

Theorem 1. (Nielson-Schreier) *If F is free and $H \leq F$, then H is free. Moreover,*

$$\mathrm{rank}(H) - 1 = (\mathrm{rank}(F) - 1)[F : H].$$

Choose and fix a prime p.

Lemma 2. *If $N \lhd G$, $[G : N] = p$, and $a \in G$, then either*

(1) *$a^G = a^N$, which holds iff $C_G(a)N = G$, or*

(2) *$a^G = a^N \cup a_2^N \cup \cdots \cup a_p^N$, which holds iff $C_G(a) \leq N$.*

Proof. The fact that $N \lhd G$ implies that G has a well defined action by conjugation on N-conjugacy classes. The G-orbit of a^N has $[G : \mathrm{Stab}(a^N)] = [G : C_G(a)N]$ N-conjugacy classes. Since N is maximal, either $C_G(a)N = G$, in which case $a^G = a^N$, or else $C_G(a) \leq N$, in which case a^G splits into p N-conjugacy classes. □

Definition 3. Let $F = F_X$ be the free group over the finite set X.

The *p-value* of an element $w \in F$ is $\|w\| = \max\{k \mid \exists v \in F(v^{p^k} = w)\}$.

The *p-deficiency* of a presentation $\langle X|R\rangle$, with X finite, is

$$\mathrm{def}_p\langle X|R\rangle = |X| - 1 - \sum_{r\in R} p^{-\|r\|}.$$

The *p-deficiency* of G is the supremum def_pG of the p-deficiencies of the presentations of G.

Lemma 4. *If G is a f.g. group with positive p-deficiency, then G has a normal subgroup N of index p.*

Proof. If G has positive p-deficiency, then it has a presentation $\langle X|R\rangle$ with positive p-deficiency. Let A_X be the elementary abelian group generated by X and let $\varphi\colon F_X \to A_X$ be the homomorphism that is the identity on generators.

Claim 5. *$\varphi(\langle R\rangle_F) \neq A_X$.*

If $r \in R$ and $\varphi(r) \neq 0$, then $\varphi(r)$ has no p-th root in A_X, so r has no p-th root in F_X. Hence $\|r\| = 0$. There must be fewer than $X - 1$ such r, since $0 < \mathrm{def}_p\langle X|R\rangle = |X| - 1 - \sum_{r\in R} p^{-\|r\|}$ forces $\sum_{r\in R} p^{-\|r\|} < |X| - 1$. This shows that $\|r\| = 0$ can hold for fewer than $|X| - 1$ elements $r \in R$. $\varphi(\langle R\rangle_F)$ is a subgroup of A generated by fewer elements than the rank of A_X, so $\varphi(\langle R\rangle_F) \neq A_X$.

Choose a maximal subgroup $M \leq A$ containing $\varphi(\langle R\rangle_F)$. $M \lhd A$ and $[A : M] = p$, so if $H = \varphi^{-1}(M)$ then $R \subseteq H \lhd F$ and $[F : H] = p$. Taking $N = H/\langle R\rangle_F$ we get $N \lhd G$ and $[G : N] = p$. □

Lemma 6. *If $N \lhd G$, $[G : N] = p$, then* $\mathrm{def}_p N \geq p \cdot \mathrm{def}_p G$.

Proof. It suffices to show that if $\langle X|R\rangle$ is a presentation for G, then there is a presentation $\langle Y|S\rangle$ for N such that $\mathrm{def}_p\langle Y|S\rangle \geq p \cdot \mathrm{def}_p\langle X|R\rangle$.

Given $\langle X|R\rangle$, let $F = F_X$ and let $\varphi\colon F \to G$ be a surjective homomorphism whose kernel is $\langle R\rangle_F$. Let $H = \varphi^{-1}(N)$, so that $\langle R\rangle_F \leq H \lhd F$, $[F : H] = p$ and $H \cong F_Y$ for some Y satisfying $|Y| - 1 = (|X| - 1) \cdot p$. The restriction of φ to H is a map from $H \cong F_Y$ onto N with kernel $\langle R\rangle_F$, so it may be used to describe a presentation of N in term of the generating set Y. We need to find the relators.

Choose $r \in R$. Let's compute $\|r\|_F$ versus $\|r\|_H$.

Case 1. $r^F = r^H$.

If $v \in F$ and $v^{p^n} = r$, then $v^p \in H$ and $(v^p)^{p^{n-1}} = v^{p^n} = r$. Thus $\|r\|_H \geq \|r\|_F - 1$. Hence $p^{-\|r\|_H} \leq p \cdot p^{-\|r\|_F}$.

Case 2. $r^F \neq r^H$.

In this case we have $r^F = r_1^H \cup r_2^H \cup \cdots \cup r_p^H$ with $r_1 = r$. If $v \in F$ is such that $v^{p^n} = r$, then $v \in C_F(r) \leq H$, so $\|r\|_F = \|r\|_H$ ($= \|r_1\|_H = \|r_2\|_H = \cdots = \|r_p\|_H$). Thus, $\sum_{i=1}^{p} p^{-\|r_i\|_H} = p \cdot p^{-\|r\|_F}$

Let $S \subseteq H$ be the set of all elements $r \in R$ for which $r^F = r^H$ holds (Case 1) together with all the elements $r_1 (= r), r_2, \ldots, r_p$ when $r^F \neq r^H$ holds (Case 2). The normal subgroup of H generated by S is the subgroup generated by the H-conjugacy classes of elements of S. This is the same as the subgroup generated by the F-conjugacy classes of elements of R, namely $\langle R\rangle_F$. Thus, $\langle Y|S\rangle$ is a presentation of N. It's p-deficiency is

$$\mathrm{def}_p\langle Y|S\rangle = |Y| - 1 - \sum_{s\in S} p^{-\|s\|_H} \geq p(|X| - 1) - \sum_{r\in R} p \cdot p^{-\|r\|_F} = p \cdot \mathrm{def}_p\langle X|R\rangle.$$

□

Lemma 7. *There is a 2-generated torsion group with positive p-deficiency.*

Proof. $G = \langle X|R\rangle$ where $X = \{x_1, x_2\}$ and $R = \{w^{p^{n(w)}} \mid w \in F_X\}$, where the numbers $n(w)$ are chosen to make the p-deficiency positive. □