

## Four Commutators

Let  $\mathbf{A}$  be an algebra.

**Definition 1.** A relation  $R \subseteq A^n$  is *compatible* if it is a subalgebra of  $\mathbf{A}^n$ . A *tolerance* is a compatible, reflexive, symmetric binary relation. A *congruence* is a compatible equivalence relation (i.e., a transitive tolerance). A *quasiorder* is a reflexive, transitive relation on  $A$ .

If  $\alpha$  and  $\beta$  are tolerances on  $\mathbf{A}$ , then

$$M(\alpha, \beta) = \left\{ \left[ \begin{array}{cc} t(\mathbf{a}, \mathbf{c}) & t(\mathbf{a}, \mathbf{d}) \\ t(\mathbf{b}, \mathbf{c}) & t(\mathbf{b}, \mathbf{d}) \end{array} \right] \mid t \text{ a term, } \mathbf{a} \alpha \mathbf{b}, \mathbf{c} \beta \mathbf{d} \right\}.$$

**Definition 2.** Let  $\alpha, \beta$  be tolerances on  $\mathbf{A}$ ,  $\delta$  be a congruence, and  $\sqsubseteq$  be a compatible quasiorder on  $A$ .

- (1)  $\mathbf{C}(\alpha, \beta; \delta)$  holds if  $p \equiv_\delta q \Rightarrow r \equiv_\delta s$  for every  $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M(\alpha, \beta)$ .
- (2)  $\mathbf{SR}(\alpha, \beta; \delta)$  holds if  $p \equiv_\delta s \Rightarrow p \equiv_\delta q \equiv_\delta r \equiv_\delta s$  for every  $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M(\alpha, \beta)$ .
- (3)  $\mathbf{S}(\alpha, \beta; \delta)$  holds if  $\mathbf{C}(\alpha, \beta; \delta)$  holds and  $\mathbf{SR}(\alpha, \beta; \delta)$  holds.
- (4)  $\mathbf{R}(\alpha, \beta; \sqsubseteq)$  holds if  $(p \sqsubseteq u \ \& \ s \sqsubseteq u) \Rightarrow (r \sqsubseteq u \ \& \ q \sqsubseteq u)$  for every  $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M(\alpha, \beta)$  and every  $u \in A$ .

**Lemma 3.** Suppose that  $\alpha$  and  $\beta$  are tolerances on  $\mathbf{A}$  and  $\mathbf{X} \in \{\mathbf{C}, \mathbf{S}, \mathbf{SR}, \mathbf{R}\}$ . The set of all relations  $\delta$  of the appropriate type (congruences or compatible quasiorders) which satisfy  $\mathbf{X}(\alpha, \beta; \delta)$  is closed under arbitrary intersection.

**Definition 4.** Let  $\alpha, \beta$  be tolerances on  $\mathbf{A}$ . If  $\mathbf{X} \in \{\mathbf{C}, \mathbf{S}, \mathbf{SR}\}$ , then  $[\alpha, \beta]_{\mathbf{X}}$  is the least congruence  $\delta$  for which  $\mathbf{X}(\alpha, \beta; \delta)$  holds. If  $\mathbf{X} = \mathbf{R}$ , then  $[\alpha, \beta]_{\mathbf{X}} = \sqsubseteq \cap \supseteq$  where  $\sqsubseteq$  is the least compatible quasiorder for which  $\mathbf{X}(\alpha, \beta; \sqsubseteq)$  holds.

$\mathbf{A}$  is *abelian* if  $[1, 1]_{\mathbf{C}} = 0$ , *strongly abelian* if  $[1, 1]_{\mathbf{S}} = 0$ , *rectangular* if  $[1, 1]_{\mathbf{R}} = 0$ , and *strongly rectangular* if  $[1, 1]_{\mathbf{SR}} = 0$ .

**Theorem 5.** If  $\mathbf{A}$  is a 2-element algebra, then

- (1)  $\mathbf{A}$  is abelian iff  $\mathbf{A}$  has a compatible Maltsev operation iff  $\mathbf{A}$  is essentially unary or polynomially equivalent to the 2-element group.
- (2)  $\mathbf{A}$  is rectangular iff  $\mathbf{A}$  has a compatible semilattice operation iff  $\mathbf{A}$  is essentially unary or polynomially equivalent to the 2-element semilattice.
- (3)  $\mathbf{A}$  is strongly abelian iff  $\mathbf{A}$  is strongly rectangular iff  $\mathbf{A}$  is essentially unary.

**Theorem 6.** If  $\mathbf{A}$  is an arbitrary algebra, then

- (1)  $\mathbf{A}$  is rectangular iff  $\mathbf{A}$  is a subalgebra of a reduct of an algebra with a compatible semilattice operation.
- (2)  $\mathbf{A}$  is strongly rectangular iff  $\mathbf{A}$  is a discretely ordered subalgebra of a reduct of an algebra with a compatible semilattice operation. (A discrete order is one in which no two distinct elements are comparable.)

**Theorem 7.** If  $\mathbf{A}$  generates a variety in which no algebra has a strongly abelian congruence, then  $\mathbf{A}$  is abelian iff  $\mathbf{A}$  is a subalgebra of a reduct of an algebra with a compatible Maltsev operation.

At the level of varieties, the following results are known. We say that “ $\mathcal{V}$  omits  $\mathbf{X}$ ” to mean that no algebra in  $\mathcal{V}$  has a nonzero congruence  $\theta$  satisfying  $[\theta, \theta]_{\mathbf{X}} = 0$ .

**Theorem 8.** *Let  $\mathcal{V}$  be a variety.*

- (A) (1)  $\mathcal{V}$  omits  $\mathbf{S}$ .  
 (2)  $\mathcal{V}$  omits  $\mathbf{SR}$ .  
 (3)  $\mathcal{V}$  satisfies a nontrivial idempotent Maltsev condition.  
 (4) There is a nontrivial identity in the language  $\{\circ, \vee, \wedge\}$  satisfied by all congruence lattices of algebras in  $\mathcal{V}$ .  
 (5)  $\mathbf{D}_1$  is not embeddable in the congruence lattice of any algebra in  $\mathcal{V}$ .  
 (6) (If  $\mathcal{V}$  is locally finite):  $\mathcal{V}$  omits type  $\mathbf{1}$ .
- (B) (1)  $\mathcal{V}$  omits  $\mathbf{R}$  (hence also  $\mathbf{SR}$ , hence also  $\mathbf{S}$ ).  
 (2)  $\mathcal{V}$  satisfies an idempotent Maltsev condition that fails in the variety of semilattices.  
 (3) There is a nontrivial identity in the language  $\{\vee, \wedge\}$  satisfied by all congruence lattices of algebras in  $\mathcal{V}$ .  
 (4)  $\mathbf{D}_2$  is not embeddable in the congruence lattice of any algebra in  $\mathcal{V}$ .  
 (5) (If  $\mathcal{V}$  is locally finite):  $\mathcal{V}$  omits types  $\mathbf{1}$  and  $\mathbf{5}$ .
- (C) (1)  $\mathcal{V}$  omits  $\mathbf{C}$  (hence also  $\mathbf{S}$ , hence also  $\mathbf{SR}$ ).  
 (2)  $\mathcal{V}$  satisfies an idempotent Maltsev condition that fails in every nontrivial variety of modules.  
 (3) The congruence lattices of algebras in  $\mathcal{V}$  satisfy
 
$$(x \wedge y = x \wedge z) \rightarrow (x \wedge y = x \wedge (y \vee z)).$$
 (4)  $\mathbf{M}_3$  is not embeddable in the congruence lattice of any algebra in  $\mathcal{V}$ .  
 (5) (If  $\mathcal{V}$  is locally finite):  $\mathcal{V}$  omits types  $\mathbf{1}$  and  $\mathbf{2}$ .
- (D) (1)  $\mathcal{V}$  omits  $\mathbf{R}$  and  $\mathbf{C}$  (hence also  $\mathbf{S}$  and  $\mathbf{SR}$ ).  
 (2) The congruence lattices of algebras in  $\mathcal{V}$  satisfy
 
$$(x \vee y = x \vee z) \rightarrow (x \vee y = x \vee (y \wedge z)).$$

