Four Commutators

Let \( A \) be an algebra.

**Definition 1.** A relation \( R \subseteq A^n \) is compatible if it is a subalgebra of \( A^n \). A tolerance is a compatible, reflexive, symmetric binary relation. A congruence is a compatible equivalence relation (i.e., a transitive tolerance). A quasiorder is a reflexive, transitive relation on \( A \).

If \( \alpha \) and \( \beta \) are tolerances on \( A \), then
\[
M(\alpha, \beta) = \left\{ \left[ \begin{array}{cc} t(a,c) & t(a,d) \\ t(b,c) & t(b,d) \end{array} \right] \mid t \text{ a term, } a \alpha b, c \beta d \right\}.
\]

**Definition 2.** Let \( \alpha, \beta \) be tolerances on \( A \), \( \delta \) be a congruence, and \( \sqsubseteq \) be a compatible quasiorder on \( A \).

1. \( C(\alpha, \beta; \delta) \) holds if \( p \equiv_\delta q \Rightarrow r \equiv_\delta s \) for every \( \left[ \begin{array}{cc} p & q \\ r & s \end{array} \right] \in M(\alpha, \beta) \).
2. \( SR(\alpha, \beta; \delta) \) holds if \( p \equiv_\delta q \Rightarrow p \equiv_\delta r \equiv_\delta s \) for every \( \left[ \begin{array}{cc} p & q \\ r & s \end{array} \right] \in M(\alpha, \beta) \).
3. \( S(\alpha, \beta; \delta) \) holds if \( C(\alpha, \beta; \delta) \) holds and \( SR(\alpha, \beta; \delta) \) holds.
4. \( R(\alpha, \beta; \sqsubseteq) \) holds if \( p \sqsubseteq u \& s \sqsubseteq u \Rightarrow (r \sqsubseteq u \& q \sqsubseteq u) \) for every \( \left[ \begin{array}{cc} p & q \\ r & s \end{array} \right] \in M(\alpha, \beta) \) and every \( u \in A \).

**Lemma 3.** Suppose that \( \alpha \) and \( \beta \) are tolerances on \( A \) and \( X \in \{ C, S, SR, R \} \). The set of all relations \( \delta \) of the appropriate type (congruences or compatible quasiorders) which satisfy \( X(\alpha, \beta; \delta) \) is closed under arbitrary intersection.

**Definition 4.** Let \( \alpha, \beta \) be tolerances on \( A \). If \( X \in \{ C, S, SR \} \), then \([\alpha, \beta]_X\) is the least congruence \( \delta \) for which \( X(\alpha, \beta; \delta) \) holds. If \( X = R \), then \([\alpha, \beta]_X = \sqsubseteq \cap \sqsubseteq \) where \( \sqsubseteq \) is the least compatible quasiorder for which \( X(\alpha, \beta; \sqsubseteq) \) holds.

**A** is abelian if \([1, 1]_C = 0\), strongly abelian if \([1, 1]_S = 0\), rectangular if \([1, 1]_R = 0\), and strongly rectangular if \([1, 1]_{SR} = 0\).

**Theorem 5.** If \( A \) is a 2-element algebra, then
1. \( A \) is abelian iff \( A \) has a compatible Mal'tsev operation iff \( A \) is essentially unary or polynomially equivalent to the 2-element group.
2. \( A \) is rectangular iff \( A \) has a compatible semilattice operation iff \( A \) is essentially unary or polynomially equivalent to the 2-element semilattice.
3. \( A \) is strongly abelian iff \( A \) is strongly rectangular iff \( A \) is essentially unary.

**Theorem 6.** If \( A \) is an arbitrary algebra, then
1. \( A \) is rectangular iff \( A \) is a subalgebra of a reduct of an algebra with a compatible semilattice operation.
2. \( A \) is strongly rectangular iff \( A \) is a discretely ordered subalgebra of a reduct of an algebra with a compatible semilattice operation. (A discrete order is one in which no two distinct elements are comparable.)

**Theorem 7.** If \( A \) generates a variety in which no algebra has a strongly abelian congruence, then \( A \) is abelian iff \( A \) is a subalgebra of a reduct of an algebra with a compatible Mal'tsev operation.
At the level of varieties, the following results are known. We say that “\( \mathcal{V} \) omits \( \mathbf{X} \)” to mean that no algebra in \( \mathcal{V} \) has a nonzero congruence \( \theta \) satisfying \( [\theta, \theta]_x = 0 \).

**Theorem 8.** Let \( \mathcal{V} \) be a variety.

(A) (1) \( \mathcal{V} \) omits \( \mathbf{S} \).
(2) \( \mathcal{V} \) omits \( \mathbf{SR} \).
(3) \( \mathcal{V} \) satisfies a nontrivial idempotent Maltsev condition.
(4) There is a nontrivial identity in the language \( \{\circ, \lor, \land\} \) satisfied by all congruence lattices of algebras in \( \mathcal{V} \).
(5) \( D_1 \) is not embeddable in the congruence lattice of any algebra in \( \mathcal{V} \).
(6) (If \( \mathcal{V} \) is locally finite): \( \mathcal{V} \) omits type 1.

(B) (1) \( \mathcal{V} \) omits \( \mathbf{R} \) (hence also \( \mathbf{SR} \), hence also \( \mathbf{S} \)).
(2) \( \mathcal{V} \) satisfies an idempotent Maltsev condition that fails in the variety of semilattices.
(3) There is a nontrivial identity in the language \( \{\lor, \land\} \) satisfied by all congruence lattices of algebras in \( \mathcal{V} \).
(4) \( D_2 \) is not embeddable in the congruence lattice of any algebra in \( \mathcal{V} \).
(5) (If \( \mathcal{V} \) is locally finite): \( \mathcal{V} \) omits types 1 and 5.

(C) (1) \( \mathcal{V} \) omits \( \mathbf{C} \) (hence also \( \mathbf{S} \), hence also \( \mathbf{SR} \)).
(2) \( \mathcal{V} \) satisfies an idempotent Maltsev condition that fails in every nontrivial variety of modules.
(3) The congruence lattices of algebras in \( \mathcal{V} \) satisfy
\[
(x \land y = x \land z) \rightarrow (x \land y = x \land (y \lor z)).
\]
(4) \( M_3 \) is not embeddable in the congruence lattice of any algebra in \( \mathcal{V} \).
(5) (If \( \mathcal{V} \) is locally finite): \( \mathcal{V} \) omits types 1 and 2.

(D) (1) \( \mathcal{V} \) omits \( \mathbf{R} \) and \( \mathbf{C} \) (hence also \( \mathbf{S} \) and \( \mathbf{SR} \)).
(2) The congruence lattices of algebras in \( \mathcal{V} \) satisfy
\[
(x \lor y = x \lor z) \rightarrow (x \lor y = x \lor (y \land z)).
\]