**Four Commutators**

Let $\mathbf{A}$ be an algebra.

**Definition 1.** A relation $R \subseteq A^n$ is *compatible* if it is a subalgebra of $\mathbf{A}^n$. A *tolerance* is a compatible, reflexive, symmetric binary relation. A *congruence* is a compatible equivalence relation (i.e., a transitive tolerance). A *quasiorder* is a reflexive, transitive relation on $A$.

If $\alpha$ and $\beta$ are tolerances on $\mathbf{A}$, then

$$M(\alpha,\beta) = \left\{ \begin{bmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{bmatrix} \;\middle|\; t \text{ a term, } \mathbf{a}\,\alpha\,\mathbf{b},\ \mathbf{c}\,\beta\,\mathbf{d} \right\}.$$

**Definition 2.** Let $\alpha, \beta$ be tolerances on $\mathbf{A}$, $\delta$ be a congruence, and $\sqsubseteq$ be a compatible quasiorder on $A$.

(1) $\mathbf{C}(\alpha,\beta;\delta)$ holds if $p \equiv_\delta q \Rightarrow r \equiv_\delta s$ for every $\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in M(\alpha,\beta)$.
(2) $\mathbf{SR}(\alpha,\beta;\delta)$ holds if $p \equiv_\delta s \Rightarrow p \equiv_\delta q \equiv_\delta r \equiv_\delta s$ for every $\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in M(\alpha,\beta)$.
(3) $\mathbf{S}(\alpha,\beta;\delta)$ holds if $\mathbf{C}(\alpha,\beta;\delta)$ holds and $\mathbf{SR}(\alpha,\beta;\delta)$ holds.
(4) $\mathbf{R}(\alpha,\beta;\sqsubseteq)$ holds if $(p \sqsubseteq u \ \&\ s \sqsubseteq u) \Rightarrow (r \sqsubseteq u \ \&\ q \sqsubseteq u)$ for every $\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in M(\alpha,\beta)$ and every $u \in A$.

**Lemma 3.** *Suppose that $\alpha$ and $\beta$ are tolerances on $\mathbf{A}$ and $\mathbf{X} \in \{\mathbf{C}, \mathbf{S}, \mathbf{SR}, \mathbf{R}\}$. The set of all relations $\delta$ of the appropriate type (congruences or compatible quasiorders) which satisfy $\mathbf{X}(\alpha,\beta;\delta)$ is closed under arbitrary intersection.*

**Definition 4.** Let $\alpha, \beta$ be tolerances on $\mathbf{A}$. If $\mathbf{X} \in \{\mathbf{C}, \mathbf{S}, \mathbf{SR}\}$, then $[\alpha,\beta]_{\mathbf{X}}$ is the least congruence $\delta$ for which $\mathbf{X}(\alpha,\beta;\delta)$ holds. If $\mathbf{X} = \mathbf{R}$, then $[\alpha,\beta]_{\mathbf{X}} = \sqsubseteq \cap \sqsupseteq$ where $\sqsubseteq$ is the least compatible quasiorder for which $\mathbf{X}(\alpha,\beta;\sqsubseteq)$ holds.

$\mathbf{A}$ is *abelian* if $[1,1]_{\mathbf{C}} = 0$, *strongly abelian* if $[1,1]_{\mathbf{S}} = 0$, *rectangular* if $[1,1]_{\mathbf{R}} = 0$, and *strongly rectangular* if $[1,1]_{\mathbf{SR}} = 0$.

**Theorem 5.** *If $\mathbf{A}$ is a 2-element algebra, then*

(1) $\mathbf{A}$ *is abelian iff* $\mathbf{A}$ *has a compatible Maltsev operation iff* $\mathbf{A}$ *is essentially unary or polynomially equivalent to the 2-element group.*
(2) $\mathbf{A}$ *is rectangular iff* $\mathbf{A}$ *has a compatible semilattice operation iff* $\mathbf{A}$ *is essentially unary or polynomially equivalent to the 2-element semilattice.*
(3) $\mathbf{A}$ *is strongly abelian iff* $\mathbf{A}$ *is strongly rectangular iff* $\mathbf{A}$ *is essentially unary.*

**Theorem 6.** *If $\mathbf{A}$ is an arbitrary algebra, then*

(1) $\mathbf{A}$ *is rectangular iff* $\mathbf{A}$ *is a subalgebra of a reduct of an algebra with a compatible semilattice operation.*
(2) $\mathbf{A}$ *is strongly rectangular iff* $\mathbf{A}$ *is a* discretely ordered *subalgebra of a reduct of an algebra with a compatible semilattice operation. (A discrete order is one in which no two distinct elements are comparable.)*

**Theorem 7.** *If $\mathbf{A}$ generates a variety in which no algebra has a strongly abelian congruence, then $\mathbf{A}$ is abelian iff $\mathbf{A}$ is a subalgebra of a reduct of an algebra with a compatible Maltsev operation.*

At the level of varieties, the following results are known. We say that "$\mathcal{V}$ omits $\mathbf{X}$" to mean that no algebra in $\mathcal{V}$ has a nonzero congruence $\theta$ satisfying $[\theta, \theta]_{\mathbf{X}} = 0$.

**Theorem 8.** *Let $\mathcal{V}$ be a variety.*

(A) (1) *$\mathcal{V}$ omits* $\mathbf{S}$.
(2) *$\mathcal{V}$ omits* $\mathbf{SR}$.
(3) *$\mathcal{V}$ satisfies a* nontrivial idempotent Maltsev condition.
(4) *There is a nontrivial identity in the language $\{\circ, \vee, \wedge\}$ satisfied by all congruence lattices of algebras in $\mathcal{V}$.*
(5) *$\mathbf{D}_1$ is not embeddable in the congruence lattice of any algebra in $\mathcal{V}$.*
(6) *(If $\mathcal{V}$ is locally finite): $\mathcal{V}$ omits type* $\mathbf{1}$.

(B) (1) *$\mathcal{V}$ omits $\mathbf{R}$ (hence also $\mathbf{SR}$, hence also $\mathbf{S}$).*
(2) *$\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in the variety of semilattices.*
(3) *There is a nontrivial identity in the language $\{\vee, \wedge\}$ satisfied by all congruence lattices of algebras in $\mathcal{V}$.*
(4) *$\mathbf{D}_2$ is not embeddable in the congruence lattice of any algebra in $\mathcal{V}$.*
(5) *(If $\mathcal{V}$ is locally finite): $\mathcal{V}$ omits types $\mathbf{1}$ and $\mathbf{5}$.*

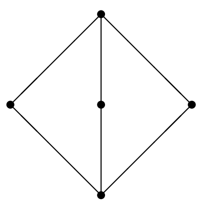
(C) (1) *$\mathcal{V}$ omits $\mathbf{C}$ (hence also $\mathbf{S}$, hence also $\mathbf{SR}$).*
(2) *$\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in every nontrivial variety of modules.*
(3) *The congruence lattices of algebras in $\mathcal{V}$ satisfy*

$$(x \wedge y = x \wedge z) \to (x \wedge y = x \wedge (y \vee z)).$$

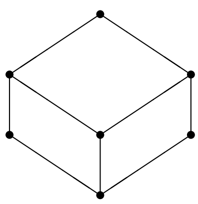
(4) *$\mathbf{M}_3$ is not embeddable in the congruence lattice of any algebra in $\mathcal{V}$.*
(5) *(If $\mathcal{V}$ is locally finite): $\mathcal{V}$ omits types $\mathbf{1}$ and $\mathbf{2}$.*

(D) (1) *$\mathcal{V}$ omits $\mathbf{R}$ and $\mathbf{C}$ (hence also $\mathbf{S}$ and $\mathbf{SR}$).*
(2) *The congruence lattices of algebras in $\mathcal{V}$ satisfy*

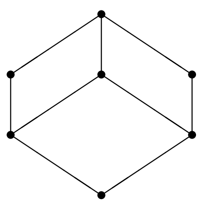
$$(x \vee y = x \vee z) \to (x \vee y = x \vee (y \wedge z)).$$



$\mathbf{M}_3$



$\mathbf{D}_1$



$\mathbf{D}_2$